PROPERTIES FOR AN INTEGRAL OPERATOR OF p-VALENT FUNCTIONS

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The purpose of this paper is to obtain new sufficient conditions for an operator on the classes of starlike and convex functions of order a and type α, the class of p-valently starlike functions, p-valently convex functions and uniformly p-valent starlike and convex functions. Refs 12.

Keywords: Analytic functions, close-to-convex functions, close-to-starlike functions, integral operator.

1. Introduction and definitions

Let $\mathcal{A}_p$ the class of all p-valent analytic functions

$$f(z) = z^p + a_{p+1}z^{p+1} + \dots, p \in \mathbb{N}$$

on the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. If we consider $p = 1$ we obtain that $\mathcal{A}_1 = \mathcal{A}$, the class of all analytical functions on $\mathcal{U}$ that satisfy the condition $f(0) = f'(0) - 1 = 0$.

We consider the classes introduced and studied by R. M. Ali and V. Ravichandran in [1].

The class of p-valent starlike functions is denoted by $\mathcal{S}_p^*(\gamma)$ and satisfy the condition

$$\frac{1}{p}\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma$$

for $\gamma < 1$ and $z \in \mathcal{U}$.

A functions $f \in \mathcal{A}_p$ is in the class of p-valent convex functions if

$$\frac{1}{p}\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \gamma,\ \gamma < 1,$$

and we denote this class by $\mathcal{K}_p$.

Starting from the classes of starlike and convex functions of complex order a and type α, R. Ali and V. Ravichandran in [1] defined the classes $S_p^*(a, \alpha)$ and $\mathcal{K}_p(a, \alpha)$ as follows:

$$\mathcal{S}_p^*(a, \alpha) = \left\{f \in \mathcal{A}_p, \alpha < 1 : \mathrm{Re}\left(1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right)\right) > \alpha\right\}$$

and

$$\mathcal{K}_p(a, \alpha) = \left\{f \in \mathcal{A}_p, \alpha < 1 : \mathrm{Re}\left(1 + \frac{1}{b}\left(\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) + 1\right)\right) > \alpha\right\}.$$

In the case of $p = 1$ the classes were studied by Breaz [4], Frasin [6], etc.

Next we will consider the classes $\mathcal{M}_p(\gamma)$ and $\mathcal{N}_p(\gamma)$.

A function $f \in \mathcal{A}_p$ is in the class $\mathcal{M}_p(\gamma)$ if

$$\frac{1}{p}\text{Re}\left(\frac{zf'(z)}{f(z)}\right) < \gamma$$

for $\gamma > 1$.

The class $\mathcal{N}_p(\gamma)$ contains all the functions that satisfy the condition

$$\frac{1}{p}\text{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) < \gamma$$

for $f \in \mathcal{A}_p$ and $\gamma > 1$.

If we consider $p = 1$, we obtain the classes $\mathcal{M}(\gamma)$ and $\mathcal{N}(\gamma)$ that were studied by many others, for example: Breaz [3], Ularu, Breaz and Frasin in [12] and Uralegaddi et al. in [11].

Also they have defined in a analogue mode the classes $\mathcal{M}_p(a, \alpha)$ and $\mathcal{N}_p(a, \alpha)$.

A function $f \in \mathcal{A}_p$ is in the class $\mathcal{M}_p(a, \alpha)$ if

$$\text{Re}\left(1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right)\right) < \alpha$$

for $\alpha > 1$.

The class $\mathcal{N}_p(a, \alpha)$ contains all the functions $f \in \mathcal{A}_p$ that satisfy

$$\text{Re}\left(1 + \frac{1}{b}\left(\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right)\right) < \alpha$$

for $\alpha > 1$.

A function f is uniformly p-valent starlike of order α with $-1 \leq \alpha < p$ in the open unit disk if

$$\text{Re}\left(\frac{zf'(z)}{f(z)} - \alpha\right) \geq \left|\frac{zf'(z)}{f(z)} - p\right|$$

for $z \in \mathcal{U}$. This class was introduced by Goodman in [7].

The class of uniformly p-valent close-to-convex functions of order α with $0 \leq \alpha < p$ in $\mathcal{U}$ contains all the functions that satisfy

$$\text{Re}\left(\frac{zf'(z)}{g(z)} - \alpha\right) \geq \left|\frac{zf'(z)}{g(z)} - p\right|$$

for $z \in \mathcal{U}$ and the function g from the class of p-valent starlike functions of order α.

To prove that our functions are p-valently starlike and p-valently close-to-convex in the open unit disk we will use the following lemmas:

Lemma 1.1 (8). *If $f \in \mathcal{A}_p$ satisfies*

$$\text{Re}\{1 + \frac{zf''(z)}{f'(z)}\} < p + \frac{1}{4} \quad \textit{for } z \in \mathcal{U}, \tag{1.1}$$

then f is p-valently starlike in $\mathcal{U}$.

Lemma 1.2 (5). *If $f \in \mathcal{A}_p$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)} + 1 - p\right| < p + 1 \quad \text{for } z \in \mathcal{U}, \tag{1.2}$$

then f is p-valently starlike in $\mathcal{U}$.

Lemma 1.3 (9). *If $f \in \mathcal{A}_p$ satisfies*

$$\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} < p + \frac{a+b}{(1+a)(1-b)} \tag{1.3}$$

for $z \in \mathcal{U}$, where $a > 0, b \geq 0$ and $a + 2b \leq 1$, then f is p-valently close-to convex in $\mathcal{U}$.

Lemma 1.4. [2] *If $f \in \mathcal{A}_p$ satisfies*

$$\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} < p + \frac{1}{3} \tag{1.4}$$

for $z \in \mathcal{U}$, then f is uniformly p-valent close-to-convex in $\mathcal{U}$.

To prove our results we consider the integral operator

$$G_{p,n}(z) = \int_0^z tp^{t-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\mu_i} \cdot \left(\frac{g_i'(t)}{pt^{p-1}}\right)^{\lambda_i} dt, \quad \mu_i, \lambda_i > 0, \tag{1.5}$$

that was studied by Stanciu and Ularu in [10].

2. Main results

Theorem 2.1. *Let $f_i, g_i \in \mathcal{A}_p$, $\mu_i, \lambda_i > 0$ and $\alpha_i, \beta_i < 1$ for $i = \overline{1,n}$. If $f_i \in \mathcal{S}_p^*(\beta_i)$ and $g_i \in \mathcal{K}_p(\alpha_i)$, then $G_{p,n} \in \mathcal{K}_p(\gamma)$, where $\gamma = 1 - \sum\limits_{i=1}^{n} \mu_i(1-\beta_i) - \sum\limits_{i=1}^{n} \lambda_i(1-\alpha_i)$.*

Proof. Starting from relation (1.5) and by logarithmic differention we obtain that

$$\frac{zG''_{p,n}(z)}{G'_{p,n}(z)} = p - 1 + \sum_{i=1}^{n} \left(\mu_i \left(\frac{zf_i'(z)}{f_i(z)} - p\right) + \lambda_i \left(\frac{zg_i''(z)}{g_i'(z)} - p + 1\right)\right).$$

It follows that

$$\frac{1}{p}\left(1 + \frac{zG''_{p,n}(z)}{G'_{p,n}(z)}\right) = \frac{1}{p}\left(p + \sum_{i=1}^{n} \left(\mu_i \left(\frac{zf_i'(z)}{f_i(z)} - p\right) + \lambda_i \left(\frac{zg_i''(z)}{g_i'(z)} - p + 1\right)\right)\right)$$

and

$$\frac{1}{p}\mathrm{Re}\left(1 + \frac{zG''_{p,n}(z)}{G'_{p,n}(z)}\right) = \frac{1}{p}\mathrm{Re}\left(p + \sum_{i=1}^{n} \left(\mu_i \left(\frac{zf_i'(z)}{f_i(z)} - p\right) + \lambda_i \left(\frac{zg_i''(z)}{g_i'(z)} - p + 1\right)\right)\right).$$

Using that $f_i \in \mathcal{S}_p^*(\beta_i)$ and $g_i \in \mathcal{K}_p(\alpha_i)$, we obtain

$$\frac{1}{p}\mathrm{Re}\left(1 + \frac{zG''_{p,n}(z)}{G'_{p,n}(z)}\right) > 1 - \sum_{i=1}^{n} \mu_i(1-\beta_i) - \sum_{i=1}^{n} \lambda_i(1-\alpha_i) = \gamma.$$

□

For $n = p = 1$ in Theorem 2.1 we obtain

Corollary 2.2. Let $f, g \in \mathcal{A}$, $\mu, \lambda > 0$ and $\alpha, \beta < 1$. If $f \in \mathcal{S}^*(\beta)$ and $g \in \mathcal{K}(\alpha)$, then $G(z) = \int_0^z t \left(\frac{f(t)}{t}\right)^{\mu} \cdot (g'(t))^{\lambda}\, dt$ is in the class $\mathcal{K}_p(\gamma)$, where $\gamma = 1 - \mu(1-\beta) - \lambda(1-\alpha)$.

If we consider $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$ in Theorem 2.1 results:

Corollary 2.3. Let $f_i, g_i \in \mathcal{A}_p$, $\mu, \lambda > 0$ and $\alpha_i, \beta_i < 1$, for $i = \overline{1,n}$. If $f_i \in \mathcal{S}_p^*(\beta_i)$ and $g_i \in \mathcal{K}_p(\alpha_i)$, then $G_{p,n}(z) = \int_0^z tp^{t-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\mu} \cdot \left(\frac{g_i'(t)}{pt^{p-1}}\right)^{\lambda} dt \in \mathcal{K}_p(\gamma)$, where $\gamma = 1 - \mu \sum_{i=1}^{n}(1-\beta_i) - \lambda \sum_{i=1}^{n}(1-\alpha_i)$.

Theorem 2.4. *Let $f_i, g_i \in \mathcal{A}_p$, $\alpha_i, \beta_i > 1$ and $\mu_i, \lambda_i > 0$ for $i = \overline{1,n}$. If $f_i \in \mathcal{M}_p(\beta_i)$ and $g_i \in \mathcal{N}_p(\alpha_i)$, then $G_{p,n}(z) \in \mathcal{N}_p(\gamma)$, where $\gamma = 1 - \sum_{i=1}^{n} \mu_i(\beta_i - 1) - \sum_{i=1}^{n} \lambda_i(\alpha_i - 1)$.*

Proof. The proof is analogue with the proof of Theorem 2.1. □

Theorem 2.5. *Let $f_i, g_i \in \mathcal{A}_p, \alpha_i, \beta_i < 1$ and $\mu_i, \lambda_i > 0$ for $i = \overline{1,n}$. If $f_i \in \mathcal{S}_p^*(a, \alpha_i)$ and $g_i \in \mathcal{K}_p(a, \beta_i)$, then $G_{p,n} \in \mathcal{K}(a, \gamma)$, where $\gamma = 1 - \sum_{i=1}^{n} \mu_i\beta_i - \sum_{i=1}^{n} \lambda_i\alpha_i$, for $i = \overline{1,n}$.*

Proof. Using the idea from the proof of Theorem 2.1 and the definition of the class $\mathcal{K}_p(a, \gamma)$ we obtain that

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{1}{a}\left(\frac{1}{p}\left(1 + \frac{zG''_{p,n}(z)}{G'_{p,n}(z)}\right) - 1\right)\right) &= \operatorname{Re}\left(1 + \frac{1}{b}\sum_{i=1}^{n}\frac{1}{p}\left(\frac{zf'(z)}{f(z)} - 1\right)\right) + \\ &+ \operatorname{Re}\left(1 + \frac{1}{b}\sum_{i=1}^{n}\lambda_i\left(\frac{1}{p}\left(\frac{zg''(z)}{g'(z)} + 1\right) - 1\right)\right) - 1 > \\ &> 1 - \sum_{i=1}^{n}\mu_i\beta_i - \sum_{i=1}^{n}\lambda_i\alpha_i. \end{aligned}$$

□

For $p = n = 1$ in Theorem 2.5 we obtain:

Corollary 2.6. Let $f, g \in \mathcal{A}, \alpha, \beta < 1$ and $\mu, \lambda > 0$. If $f \in \mathcal{S}^*(a, \alpha)$ and $g \in \mathcal{K}(a, \beta)$, then $G(z) = \int_0^z t \left(\frac{f(t)}{t}\right)^{\mu} \cdot (g'(t))^{\lambda}\, dt \in \mathcal{K}(a, \gamma)$, where $\gamma = 1 - \mu\beta - \lambda\alpha$.

Theorem 2.7. *Let μ_i, λ_i positive real numbers and $f_i, g_i \in \mathcal{A}_p$ for $i = \overline{1,n}$. If f_i satisfies*

$$\operatorname{Re}\frac{zf_i'(z)}{f_i(z)} < p + \frac{1}{\sum_{i=1}^{n}\mu_i}$$

and g_i satisfies

$$\operatorname{Re}\left(\frac{zg_i''(z)}{g_i'(z)} + 1\right) < p - \frac{3}{4\sum_{i=1}^{n}\lambda_i},$$

then $G_{p,n}$ is p-valently starlike in the open unit disk.

Proof. From the definition of $G_{p,n}$ and from the hypotheses of our theorem it follows:

$$\operatorname{Re}\{1+\frac{zG''_{p,n}(z)}{G'_{p,n}(z)}\}=p\left(1-\sum_{i=1}^{n}\mu_i-\sum_{i=1}^{n}\lambda_i\right)+\sum_{i=1}^{n}\operatorname{Re}\mu_i\frac{zf'_i(z)}{f_i(z)}+\sum_{i=1}^{n}\operatorname{Re}\lambda_i\left(\frac{zg''_i(z)}{g'_i(z)}+1\right)<$$

$$<p\left(1-\sum_{i=1}^{n}\mu_i-\sum_{i=1}^{n}\lambda_i\right)+\sum_{i=1}^{n}\mu_i\left(p+\frac{1}{\sum\limits_{i=1}^{n}\mu_i}\right)+\sum_{i=1}^{n}\lambda_i\left(p-\frac{3}{4\sum\limits_{i=1}^{n}\lambda_i}\right)<$$

$$<p+\frac{1}{4}.$$

According to Lemma 1.1 we obtain that $G_{p,n}$ is in the class of p-valently starlike functions. □

If in Theorem 2.7 we consider $n=p=1$, then we obtain:

Corollary 2.8. Let μ,λ positive real numbers and $f,g\in\mathcal{A}$. If f satisfies

$$\operatorname{Re}\frac{zf'(z)}{f(z)}<1+\frac{1}{\mu}$$

and g satisfies

$$\operatorname{Re}\left(\frac{zg''(z)}{g'(z)}+1\right)<1-\frac{3}{4\lambda},$$

then $G(z)=\int_0^z t\left(\frac{f(t)}{t}\right)^{\mu}\cdot(g'(t))^{\lambda}\,dt$ is starlike in the open unit disk.

Theorem 2.9. *Let μ_i,λ_i positive real numbers and $f_i,g_i\in\mathcal{A}_p$ for $i=\overline{1,n}$. If the functions f_i satisfies*

$$\left|\frac{zf'_i(z)}{f_i(z)}-p\right|<\frac{p}{\sum\limits_{i=1}^{n}\mu_i}$$

and the functions g_i satisfies

$$\left|\frac{zg''_i(z)}{g'_i(z)}-p+1\right|<\frac{1}{\sum\limits_{i=1}^{n}\lambda_i}$$

for $z\in\mathcal{U}$, then $G_{p,n}$ is p-valently starlike in $\mathcal{U}$.

Proof. Using the hypotheses of these theorem, we obtain that

$$\left|\frac{zG''_{p,n}(z)}{G'_{p,n}(z)}+1-p\right|=\left|\sum_{i=1}^{n}\mu_i\left(\frac{zf'_i(z)}{f_i(z)}-p\right)+\lambda_i\left(\frac{zg''_i(z)}{g'_i(z)}-p+1\right)\right|<$$

$$<\sum_{i=1}^{n}\mu_i\left|\frac{zf'_i(z)}{f_i(z)}-p\right|+\sum_{i=1}^{n}\lambda_i\left|\frac{zg''_i(z)}{g'_i(z)}-p+1\right|<$$

$$<p+1.$$

Using Lemma 1.2, results that $G_{p,n}$ is p-valently starlike in $\mathcal{U}$. □

For $p = n = 1$ in Theorem 2.9 results:

Corollary 2.10. Let μ, λ positive real numbers and $f, g \in \mathcal{A}$. If the functions f satisfies

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{\mu}$$

and the functions g satisfies

$$\left|\frac{zg''(z)}{g'(z)}\right| < \frac{1}{\lambda}$$

for $z \in \mathcal{U}$, then $G(z) = \int_0^z t\left(\frac{f(t)}{t}\right)^{\mu} \cdot (g'(t))^{\lambda}\, dt$ is starlike in $\mathcal{U}$.

Theorem 2.11. *Let μ_i, λ_i positive real numbers and $f_i, g_i \in \mathcal{A}_p$ for $i = \overline{1,n}$. If f_i satisfies*

$$\mathrm{Re}\frac{zf_i'(z)}{f_i(z)} < p + \frac{a}{(1+a)(1-b)\sum\limits_{i=1}^{n}\mu_i},$$

and g_i satisfies

$$\mathrm{Re}\left(1 + \frac{zg_i''(z)}{g_i'(z)}\right) < p + \frac{b}{(1+a)(1-b)\sum\limits_{i=1}^{n}\lambda_i}$$

for $a > 0, b \geq 0, a + 2b \leq 1$ and $z \in \mathcal{U}$, then $G_{p,n}$ is p-valently close-to-convex in $\mathcal{U}$.

Proof. The proof is similar with the proof of Theorem 2.7, but we use Lemma 1.3. □

Considering $p = n = 1$ in Theorem 2.11 results:

Corollary 2.12. Let μ, λ positive real numbers and $f, g \in \mathcal{A}$. If f satisfies

$$\mathrm{Re}\frac{zf'(z)}{f(z)} < 1 + \frac{a}{(1+a)(1-b)\mu}$$

and g satisfies

$$\mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) < 1 + \frac{b}{(1+a)(1-b)\lambda}$$

for $a > 0, b \geq 0, a + 2b \leq 1$ and $z \in \mathcal{U}$, then $G(z) = \int_0^z t\left(\frac{f(t)}{t}\right)^{\mu} \cdot (g'(t))^{\lambda}\, dt$ is close-to-convex in $\mathcal{U}$.

Theorem 2.13. *Let μ_i, λ_i positive real numbers and $f_i, g_i \in \mathcal{A}_p$ for $i = \overline{1,n}$. If f_i satisfies*

$$\mathrm{Re}\frac{zf_i'(z)}{f_i(z)} < p + \frac{1}{\sum\limits_{i=1}^{n}\mu_i}$$

and g_i satisfies

$$\mathrm{Re}\left(\frac{zg_i''(z)}{g_i'(z)} + 1\right) < p - \frac{2}{3\sum\limits_{i=1}^{n}\lambda_i}$$

for $z \in \mathcal{U}$, then $G_{p,n}$ is uniformly p-valently close-to-convex in $\mathcal{U}$.

Proof. The proof is similar to Theorem 2.7, using Lemma 1.4. □

For $p = n = 1$ in Theorem 2.13 we obtain:

Corollary 2.14. Let μ, λ positive real numbers and $f, g \in \mathcal{A}$. If f satisfies

$$\mathrm{Re}\frac{zf'(z)}{f(z)} < 1 + \frac{1}{\mu}$$

and g satisfies

$$\mathrm{Re}\left(\frac{zg''(z)}{g'(z)} + 1\right) < 1 - \frac{2}{3\lambda}$$

for $z \in \mathcal{U}$, then $G(z) = \int_0^z t\left(\frac{f(t)}{t}\right)^{\mu} \cdot (g'(t))^{\lambda}\, dt$ is uniformly close-to-convex in $\mathcal{U}$.

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СВОЙСТВА ИНТЕГРАЛЬНОГО ОПЕРАТОРА p-ЗНАЧНЫХ ФУНКЦИЙ

Николета Улау, Лаура Станчи

Целью статьи является получение новых достаточных условий для оператора на классах звездообразных и выпуклых функций порядка a и типа α, класса p-значных звездообразные функций, p-значных выпуклых функций и однородно p-значных звездообразных и выпуклых функций. Библиогр. 12 назв.

Ключевые слова: аналитические функции, функции близкие к выпуклым, функции близкие к звездообразным, интегральный оператор.