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SOME PROPERTIES OF EXACT RUIN PROBABILITY FOR ERLANG-DISTRIBUTED CLAIMS

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Мы обсуждаем аналитическую структуру точного выражения для вероятности разорения в классической модели Крамера—Лундберга с исками, распределенными по Эрлангу. Эта формула может быть легко получена стандартным методом преобразования Лапласа. Однако ее тонкая структура зависит существенным образом от отсутствия (наличия) кратных корней у некоторого полиномиального уравнения. Мы доказываем, что все корни этого уравнения простые (имеют кратность один). Как следствие, вероятность разорения представляется как линейная комбинация чистых экспонент (в общем случае, комплексных экспонент). Библиогр. 3 назв.

Ключевые слова: вероятность разорения, иски, распределенные по Эрлангу.

Classical Cramèr—Lundberg model (see for instance [1–3]) studies the ruin probability for stochastic process

$$X(t) = u + ct - \sum_{i=1}^{N(t)} Y_i.$$

Here $N(t)$ is a homogeneous Poisson process of intensity λ, $Y_i, i \geq 1$, are positive mutually independent identically distributed random variables (amounts of claims) independent of the process $N(t)$. We'll suppose that $a = \mathbf{E}Y_i < \infty$ and that $\rho = (c - a\lambda)/(a\lambda) > 0$ (ρ is the so called safety loading).

In fact we'll study the special case of Erlang-distributed claims. Erlang distribution is a continuous distribution with probability density function

$$f(z) = \frac{b^n}{(n-1)!} z^{n-1} e^{-bz}, \quad z > 0$$

and moment generating function

$$M(r) = \left(\frac{b}{b-r}\right)^n.$$

Here $a = n/b$. The moment generating function is defined for $r < b$. Obviously Erlang distribution is nothing but Gamma distribution with integer shape parameter $n \geq 1$.

Let

$$\Phi(u) = \mathbf{P}((\forall t \geq 0)(X(t) \geq 0))$$

be a non-ruin probability. As tail probabilities are exponentially small for Erlang distribution we can use the following formula for Laplace transform

$$\int_0^\infty e^{ru} \Phi'(u) dx = \frac{\rho}{1+\rho} \cdot \frac{\frac{M(r)-1}{r}}{a(1+\rho) - \frac{M(r)-1}{r}}$$

(see [2], p. 13) valid for sufficiently small positive r.

The right hand side of this equality is in Erlang case the rational algebraic fraction. So to invert the Laplace transform in explicit form it is sufficient to represent this right hand side as a sum of elementary fractions. For this we need to study some polynomial equation ($\equiv$ to study the roots of a denominator):

$$\left(\frac{b}{b-r}\right)^n = a(1+\rho)r + 1. \tag{1}$$

We'll prove that this equation has (in addition to trivial $r = 0$) n distinct roots $R_1, \ldots, R_n$. For n odd only one of them is real and for n even only two of them are real. It follows easily from (1) that real parts of these roots are positive. So our result gives the following expressions

$$\Phi'(u) = \sum_{i=1}^{n} c_i e^{-R_i u},$$

$$\Phi(u) = 1 - \sum_{i=1}^{n} \frac{c_i}{R_i} e^{-R_i u}.$$

To prove our statement concerning roots let denote

$$y = \frac{b}{b-r}.$$

Then y satisfies the equation

$$f(y) = 0, \tag{2}$$

where

$$f(y) = y^{n+1} - (n+1+n\rho)y + n + n\rho$$

(we take into account the equality $ab = n$). This equation has the same (up to the change of variable) roots as (1) including additional trivial root $y = 1$ corresponding to $r = 0$.

We check now that polynomials $f(y)$ and $f'(y) = (n+1)y^n - (n+1+n\rho)$ have no common roots. Indeed, $f'(y) = 0$ for

$$y_k = \left(1 + \frac{n}{n+1}\rho\right)^{\frac{1}{n}} \left(\cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right)\right),$$

$(k = 0, 1, \ldots, n-1)$. Then

$$f(y_k) = y_k \left(1 + \frac{n}{n+1}\rho - n - 1 - n\rho\right) + n + n\rho.$$

This expression is real for $k = 0$ and $k = n/2$ (in the case of even n)

Suppose firstly y_0 be a root of polynomial $f(y)$. Then its multiplicity needs to be 2 (as $f'(y_0) = 0$). On the other hand $f'(y) > 0$ for positive y and f is strictly increasing for such y. This contradicts to previous conclusion about multiplicity.

Suppose secondly that $n = 2m$. Then

$$y_m = -\left(1 + \frac{2m}{2m+1}\rho\right)^{1/(2m)} < -1.$$

Dividing (2) on $y-1$ we receive the equation

$$y^n + y^{n-1} + \ldots + y = n(1+\rho).$$

Denoting $z = -y$ we can write it as

$$(z^{2m} - z^{2m-1}) + \ldots + (z^2 - z) = 2m(1+\rho). \tag{3}$$

Our task is to prove that

$$\tilde{z} = -y_m = \left(1 + \frac{2m}{2m+1}\right)^{1/(2m)}$$

cannot be a root of this equation (especially to be a root of multiplicity 2). Indeed writing the left hand side of (3) as

$$(z-1)(z^{2m-1} + z^{2m-3} + \ldots + z)$$

we note that this expression is strictly increasing for $z > 1$. So $\tilde{z}$ cannot be a root of multiplicity 2.

We already received that equations (1) and (2) have no multiple roots. It rests to prove the statement about the number of real roots. Taking into account the positivity of such roots we need to analyse two cases: 1) $0 < r < b$; 2) $n = 2m$, $r > b$.

In the case 1) there is exactly one root of (1) — convexity of power function. In the case 2) also there is exactly one root — opposite directions of monotonicity for left and right parts of (1).

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We discuss analytic structure of exact expression for ruin probability in classical Cramèr—Lundberg model with Erlang-distributed claims. This formula can be easily received via standard Laplace transform method. Nevertheless its fine structure depends crucially on the absence (presence) of multiple roots of some polynomial equation. We prove that all roots of this equation are simple (have multiplicity one). As a consequence the ruin probability represents as a linear combination of pure exponents (complex exponents in general case). Refs 3.

Keywords: ruin probability, Erlang-distributed claims.