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THE FIRST INITIAL-BOUNDARY VALUE PROBLEM FOR PARABOLIC EQUATIONS IN A CONE WITH EDGES

Vu Trong Luong[1], *Do Van Loi*[2]

[1] Department of Mathematics, Taybac University,
Sonla city, Sonla, Vietnam
[2] Department of Mathematics, Hongduc University,
Thanhhoa, Vietnam

The initial-boundary value problem with Dirichlet boundary condition for higher order parabolic equations in a cone with edges is considered. We prove the well-posedness by the similar arguments as in [19]. Moreover, the regularity of the solution are also proven. Refs 25.

Keywords: well-posedness, initial-boundary value problems, regularity, cone with edges.

1. Introduction. Let $\mathcal{K} = \{x \in \mathbb{R}^3 : \ x/|x| = \omega \in \Omega\}$ be open polyhedral domain in $\mathbb{R}^3$ with vertex at the origin. Suppose that the boundary $\partial\mathcal{K}$ consists of the vertex $x = 0$, the edges (half-lines) $M_1, \cdots, M_d$, and smooth (of class C^∞) faces $\Gamma_1, \cdots, \Gamma_d$. This means that $\Omega = \mathcal{K} \cap S^2$ is a domain of polygonal type on the unit sphere S^2 with sides $\gamma_k = \Gamma_k \cap S^2$. Let $T, 0 < T < \infty$, and $\Gamma_{j,T} = \Gamma_j \times (0,T),\ j = 1, \cdots, d; \mathcal{K}_T = \mathcal{K} \times (0,T)$.

Let the partial differential operator given by

$$L(x,t;D) = \sum_{|p|,|q|=0}^{m} D_x^p(a_{pq}(x,t)D_x^q),$$

where a_{pq} are bounded functions with complex values from $C^\infty(\mathcal{K}_T)$, $a_{pq} = (-1)^{p+q}\overline{a}_{qp}$ and $\overline{a}_{qp}$ denotes the conjugate of a_{qp}.

We also suppose that operator L is strong elliptic uniformly with respect to $t \in [0,T)$, that is, there exists a constant $c > 0$ such that

$$\sum_{|p|,|q|=m} a_{pq}\xi^p\xi^q \geq c|\xi|^{2m}, \quad \forall (x,t) \in \mathcal{K}_T \tag{1.1}$$

forall vector $\xi \in \mathbb{R}^n$.

Consider the initial-boundary value problem

$$u_t + (-1)^m L(x,t;D_x)u = f \quad \text{in } \mathcal{K}_T, \tag{1.2}$$

$$\frac{\partial^{k-1}u}{\partial\nu^{k-1}}\Big|_{\Gamma_{j,T}} = 0, \quad k = 1, \cdots, m, \quad j = 1, \cdots, d, \tag{1.3}$$

$$u|_{t=0} = 0 \quad \text{in } \mathcal{K}. \tag{1.4}$$

Here function $f(x,t)$ is given on $\mathcal{K}_T, \nu$ denotes the exterior normal to $\Gamma_{jT}, j = 1, \cdots, d$.

Elliptic boundary value problems in polyhearal domains have been studied by Maz'ya and Rossman in the monograph [22]. Along with elliptic boundary value problems, mathematicians have paid considerable attention to initial-boundary value problems for parabolic equations in domains with conical points or with edges. In [7, 8] Maz'ya and Kozlov considered the heat equation in domains with conical points in which the asymptotics of the solutions near conical points was studied. For domains with edges, Solonnikov [24, 25] and Nazarov [23] estimated the Green function and proved the existence of solutions of the Dirichlet and Neumann problems for the heat equation in weighted

Sobolev spaces. In [13, 14] Kozlov and Rossman have been studied the asymptotics of the solutions of the Dirichlet problem for the heat equation near an edge. In [9–11] Kozlov has been dealt with for general second order parabolic equations with time-independent coefficients in domains with conical points, where the asymptotics of solutions and a description of the sigularities of the Green function near the conical points were obtained. In the case of time-dependent coefficients, let us mention some works related to this case. In [1–4] in which the unique existence of weak solutions in W_p^1-Sobolev spaces was established. In [6] one investigated results on the existence, uniqueness and regularity of generalized solution of equation (1.2) with initial and general boundary conditions in conical domains. Recently, in [19] and [20] we considered the Cauchy—Dirichlet problem for nonstationary equations of second order in domains with edges.

In contrast to the above papers, in this work, we consider higher order parabolic equations with time-dependent coefficients in a cone with edges. By modifying the method suggested in [19] to obtain the well-posedness of problem (1.2)–(1.4). Furthemore, we prove the regularity of the solution in weighted Sobolev spaces with using the help of regularity results for elliptic boundary value problems in [22].

2. The well-posedness of the problem. Fistly, we will introduce some Sobolev spaces as usual on $\mathcal{K}$ and $\mathcal{K}_T$.

1. $H^m(\mathcal{K})$ is a Sobolev space complex functions $u(x)$ defined on $\mathcal{K}$ with the norm
$$\|u\|_{H^m(\mathcal{K})} = \Big(\sum_{|p|\le m} \int_{\mathcal{K}} |D^p u|^2 dx\Big)^{\frac{1}{2}} < +\infty.$$
2. $\overset{\circ}{H}{}^m(\mathcal{K})$ denotes the closure of $C_0^\infty(\mathcal{K})$ in $H^m(\mathcal{K})$.
3. $H^{m,h}(\mathcal{K}_T)$ denotes the Sobolev space complex functions $u(x,t)$ defined on $\mathcal{K}_T$ with the norm
$$\|u\|_{H^{m,h}(\mathcal{K}_T)} = \Big(\int_{\mathcal{K}_T} \Big(\sum_{|p|\le m} |D^p u|^2 + \sum_{j=1}^{h} |u_{t^j}|^2\Big) dxdt\Big)^{\frac{1}{2}} < +\infty,$$
where $p=(p_1,\cdots,p_n); m,k$ are nonnegative integers.
4. The space $\overset{\circ}{H}{}^{m,k}(\mathcal{K}_T)$ is the closure in $H^{m,k}(\mathcal{K}_T)$ of the set consiting of all functions $u\in C^\infty(\mathcal{K}_T)$, which vanish near $\partial\mathcal{K}_T=\bigcup_{j=1}^d \overline{\Gamma}_{j,T}$.

Let us denote by
$$B(u,v;t) = \int_{\mathcal{K}} \sum_{|p|,|q|=0}^{m} (-1)^{|p|} a_{pq}(x,t) D_x^q u \overline{D_x^p v} dx$$
the time-dependent bilinear form. Then, we have the following Green's formula:
$$(L(x,t;D)u,v)_{L_2(\mathcal{K})} = B(u,v;t),$$
which is valid for all $u,v\in C_0^\infty(\mathcal{K})$ and a.e. $t\in[0,T]$.

Definition 2.1. *A function $u \in \overset{\circ}{H}{}^{m,1}(\mathcal{K}_T)$ is called a generalized solution of problem (1.2)–(1.4), if and only if $u(x,0) = 0,\ \forall x \in \mathcal{K}$ and the equality*

$$(u_t, v)_{L_2(\mathcal{K})} + (-1)^m B(u, v; t) = (f, v)_{L_2(\mathcal{K})},\ \text{a.e. } t \in [0, T], \tag{2.1}$$

holds for all $v \in \overset{\circ}{H}{}^m(\mathcal{K})$.

From the assumptions above, we also have the Garding's inequality, i.e., there exist constants $\mu_0 > 0, \lambda_0 \geq 0$ such that

$$(-1)^m B(u, u; t) \geq \mu_0 \|u\|^2_{H^m(\mathcal{K})} - \lambda_0 \|u\|^2_{L_2(\mathcal{K})} \tag{2.2}$$

holds for all $u \in \overset{\circ}{H}{}^m(\mathcal{K})$ and a.e. $t \in [0, T]$.

We note that the constant λ_0 can be chosen with 0, since by a substitution $v = e^{-\lambda_0 t} u$ the operator L can be transformed to $\widetilde{L} = L + \lambda_0$, with the time-dependent bilinear form associated with $\widetilde{L}$ is $\widetilde{B}(.,.;t)$ satisfying (2.2) with the constant $\lambda_0 = 0$. Hence, throughout the present paper we also suppose that $B(.,.;t)$ satisfying the following inequality:

$$(-1)^m B(u, u; t) \geq \mu_0 \|u\|^2_{H^m(\mathcal{K})} \tag{2.3}$$

for all $u \in \overset{\circ}{H}{}^m(\mathcal{K})$ and a.e. $t \in [0, T]$.

By Galerkin's approximating method and arguments similar as in [19], we have the following theorem.

Theorem 2.1. *Let $f \in L_2(\mathcal{K}_T)$, and suppose that the coefficients of the operator L satisfy*

$$\sup\{|a_{pq}|, |a_{pqt}| : (x,t) \in \mathcal{K}_T\} \leq \mu, \quad \mu = \text{const}.$$

Then problem (1.2)–(1.4) has unique generalized solution u in the space $\overset{\circ}{H}{}^{m,1}(\mathcal{K}_T)$ and the following estimate holds:

$$\|u\|^2_{H^{m,1}(\mathcal{K}_T)} \leq C \|f\|^2_{L_2(\mathcal{K}_T)}, \tag{2.4}$$

here C is a constant independent of u and f. This solution depends continuously on f.

The results above shows the unique solvability of problem (1.2)–(1.4). Furthemore, the next observation shows that the generalized solution dependens continuously on the right-hand side f of (1.2).

Now we will prove the continuous dependence on the coefficients of the operator L. Let $\delta > 0$, we denote by

$$L^\delta = L^\delta(x, t; D) := \sum_{0 \leq |p|, |q| \leq m} D^p(a^\delta_{pq}(x,t) D^q),$$

the operator depends on δ, the coefficients a^δ_{pq} are bounded functions with complex values from $C^\infty(\mathcal{K}_T)$, $a^\delta_{pq} = (-1)^{|p|+|q|} a^{\delta *}_{qp}, a^{\delta *}_{qp}$ denotes the transposed conjugate matrix of a^δ_{qp}. Set

$$B^\delta(u, v; t) = \int_\Omega \sum_{|p|,|q|=0}^{m} (-1)^{|p|} a^\delta_{pq}(x,t) D^q_x u \overline{D^p_x v} dx,$$

assume also that the Garding's inequality

$$(-1)^m B^\delta(u, u; t) \geq \widehat{\mu}_0 \|u\|^2_{H^m(\mathcal{K})}, \quad \widehat{\mu}_0 > 0, \tag{2.5}$$

holds for all $u \in \overset{\circ}{H}{}^m(\mathcal{K})$ and a.e. $t \in [0,T]$. Let u^δ be the generalized solution of problem (1.2)–(1.4) with replacing the operator L by L^δ. Then we have the following theorem.

Theorem 2.2. *Let u be the generalized solution of problem* (1)–(3). *Suppose that*

$$\sup\{|a_{pq}(x,t) - a^\delta_{pq}(x,t)| : 0 \le |p|, |q| \le m, (x,t) \in \mathcal{K}_T\} \le \theta(\delta) \to 0 \quad as \quad \delta \to 0.$$

Then functions u^δ conveges to u as $\delta \to 0$.

Proof. By setting $U^\delta = u^\delta - u$, we get from (2.1) that

$$(U^\delta_t, v) + (-1)^m B^\delta(U^\delta, v; t) = \sum_{0 \le |p|,|q| \le m} (-1)^{|p|+m}(a_{pq} - a^\delta_{pq}) D^q u \overline{D^p v} dx \tag{2.6}$$

holds for all $v \in \overset{\circ}{H}{}^m(\mathcal{K})$. Let $\{\omega_k(x)\}_{k=1}^\infty$, as in Theorem 2.1, set $U^{\delta,N}(x,t) = \sum_{k=1}^N C^{\delta,N}_k(t)\omega_k(x)$, with $\{C^{\delta,N}_k\}_{k=1}^N$ are the solution of the system of the following ordinary differential equations

$$\begin{aligned}(U^{\delta,N}_t, \omega_k) &+ (-1)^m B(U^{\delta,N}, \omega_k; t) = \\ &= \sum_{0 \le |p|,|q| \le m} (-1)^{|p|+m}(a_{pq} - a^\delta_{pq}) D^q u \overline{D^p \omega_k} dx, \quad t \in [0,T), \quad k = 1, \dots, N,\end{aligned} \tag{2.7}$$

with the initial conditions

$$C^{\delta,N}_k(0) = 0, k = 1, \dots, N. \tag{2.8}$$

Let us multiply (2.7) by $C^{\delta,N}_k(t)$, sum $k = 1, \dots N$, to find

$$\begin{aligned}(U^{\delta,N}_t, U^{\delta,N}) &+ (-1)^m B(U^{\delta,N}, U^{\delta,N}; t) = \\ &= \sum_{0 \le |p|,|q| \le m} (-1)^{|p|+m}(a_{pq} - a^\delta_{pq}) D^q u \overline{D^p U^{\delta,N}} dx.\end{aligned} \tag{2.9}$$

Now adding this equality to its complex conjugate, we get

$$\begin{aligned}\frac{d}{dt}\big(\|U^{\delta,N}\|^2_{L_2(\mathcal{K})}\big) &+ (-1)^m 2B(U^{\delta,N}, U^{\delta,N}; t) = \\ &= 2\mathrm{Re} \sum_{0 \le |p|,|q| \le m} (-1)^{|p|+m}(a_{pq} - a^\delta_{pq}) D^q u \overline{D^p U^{\delta,N}} dx.\end{aligned} \tag{2.10}$$

Employing inequality (2.5) and the Cauchy inequality, we obtain from (2.10) the estimate

$$\frac{d}{dt}\big(\|U^{\delta,N}\|^2_{L_2(\mathcal{K})}\big) + (2\widehat{\mu}_0 - \varepsilon)\|U^{\delta,N}\|^2_{H^m(\mathcal{K})} \le C\theta(\delta)\|u\|^2_{H^m(\mathcal{K})}, \tag{2.11}$$

where C only depens on ε. Choosing $0 < \varepsilon < 2\widehat{\mu}_0$, we have

$$\frac{d}{dt}\big(\|U^{\delta,N}\|^2_{L_2(\mathcal{K})}\big) \le C\theta(\delta)\|u\|^2_{H^m(\mathcal{K})}. \tag{2.12}$$

Integrating them with respect to t from 0 to $\tau, \tau \in (0,T)$, we obtain

$$\int_0^\tau \left(\frac{d}{dt}\|U^{\delta,N}\|^2_{L_2(\mathcal{K})}\right) dt \le C\theta(\delta)\|u\|^2_{H^{m,0}(\mathcal{K}_T)} \le C\theta(\delta)\|f\|_{L_2(\mathcal{K}_T)}.$$

By arguments analogous to the proof of Theorem 2.1 of [19], we arrive at

$$\|U^{\delta,N}\|^2_{H^{m,1}(\mathcal{K}_T)} \le C\theta(\delta)\|f\|_{L_2(\mathcal{K}_T)}.$$

Therefore,

$$\|U^{\delta}\|^2_{H^{m,1}(\mathcal{K}_T)} \le \liminf_{N\to\infty} \|U^{\delta,N}\|^2_{H^{m,1}(\mathcal{K}_T)} \le C\theta(\delta)\|f\|_{L_2(\mathcal{K}_T)}.$$

It means that $\|U^{\delta}\|^2_{H^{m,1}(\mathcal{K}_T)} \to 0$ as $\delta \to 0$. The theorem is proved.

3. The regularity of the generalized solution. In this section, we discuss the regularity of the generalized solution u of problem (1.2)–(1.4). Firstly, we give a needed auxiliary lemma, which deal with the regularity the solution with respect to time variable. It is proved by repeating almost word for word in the proof of Theorem 3.1 of [6].

Lemma 3.1. *Let $h \in \mathbb{N}^*$, and we assume that*

(i) $\sup\big\{|a_{pqt^k}| : i,j = 1,\dots,n; (x,t) \in \mathcal{K}_T, k \le h+1\big\} \le \mu,$

(ii) $f_{t^k} \in L_2(\mathcal{K}_T), \quad k \le h; \quad f_{t^k}(x,0) = 0, \quad 0 \le k \le h-1.$

Then the generalized solution $u \in \mathring{H}^{m,1}(\mathcal{K}_T)$ of problem (1.2)–(1.4) has derivatives with respect to t up to order h with $u_{t^k} \in \mathring{H}^{m,1}(\mathcal{K}_T), k = 0,\dots,h$, and

$$\|u_{t^h}\|^2_{H^{m,1}(\mathcal{K}_T)} \le C \sum_{j=0}^{h} \|f_{t^j}\|^2_{L_2(\mathcal{K}_T)}, \tag{3.1}$$

where C is a constant independent of u and f.

Next, we will show the global regularity of the solution. To do this, we introduce operator pencils generated by the Dirichlet problem for elliptic equation in cone $\mathcal{K}$. Let M_k be an edge of the cone $\mathcal{K}$, and let $\Gamma_{k_+}, \Gamma_{k_-}$ be the faces adjacent to M_k. Then by $\mathcal{D}_k$ we denote the dihedron which is bounded by the half-planes $\Gamma^\circ_{k_\pm}$ tangent to $\Gamma_{k_\pm}$ at M_k. Let r, φ be polar coordinates in the plane perpendicular to M_k such that

$$\Gamma^\circ_{k_\pm} = \{x \in \mathbb{R}^3 : r > 0, \varphi = \pm\theta_k/2\}.$$

Fix $t \in [0,T]$, we define the operator $A_k(\lambda,t)$ as follows:

$$A_k(\lambda,t)U = r^{2m-\lambda}L^0(0,t,D)(r^\lambda U),$$

where $L^0(0,t,D) = \sum\limits_{|p|=|q|=m} D^p(a_{pq}(0,t)D^q)$, $u(x) = r^\lambda U(\varphi)$, $\lambda \in \mathbb{C}$. The operator $A_k(\lambda,t)$ realizes a continuous mapping from $W^{2m}_2(I_k) \cap \mathring{W}^m_2(I_k)$ into $L_2(I_k)$ for every $\lambda \in \mathbb{C}$, where I_k denotes the interval $(-\theta_k/2, \theta_k/2)$. A complex number λ_0 is called an eigenvalue of the pencil $A_k(\lambda,t)$ if there exists a nonzero function $U \in W^{2m}_2(I_k) \cap \mathring{W}^m_2(I_k)$ such that $A_k(\lambda_0,t)U = 0$. We denote by $\delta^{(k)}_+(t)$ and $\delta^{(k)}_-(t)$ the greatest positive real numbers such that the strip

$$m - 1 - \delta^{(k)}_-(t) < \mathrm{Re}\lambda < m - 1 + \delta^{(k)}_+(t)$$

is free of eigenvalues of the pencil $A_k(\lambda,t)$. Furthemore, we define

$$\delta^{(k)}_\pm = \inf_{t\in[0,T]} \delta_\pm(t)$$

for $k = 1,\dots,d$.

We introduce spherical coordinates $\rho = |x|, \omega = x/|x|$ in $\mathcal{K}$ and define

$$\mathfrak{U}(\lambda, t)U = \rho^{2m-\lambda} L^0(0, t, D)(\rho^\lambda U),$$

where $u(x) = \rho^\lambda U(\omega)$. The operator $\mathfrak{U}(\lambda, t)$ realizes a continuous mapping

$$W_2^{2m}(\Omega) \cap \mathring{W}_2^m(\Omega) \to L_2(\Omega).$$

An eigenvalue of $\mathfrak{U}(\lambda, t)$ is a complex number λ_0 such that $\mathfrak{U}(\lambda_0, t)U = 0$ for some nonzero function $U \in W_2^{2m}(\Omega) \cap \mathring{W}_2^m(\Omega)$.

Let l be a nonnegative integer, $\beta \in \mathbb{R}, \delta = (\delta_1, \cdots, \delta_d) \in \mathbb{R}^d$. Furthermore, let $S = \{0\} \cup M_1 \cup \cdots \cup M_d$ be the set of the singular boundary points. Then $V^l_{\beta,\delta}(\mathcal{K})$ is defined as the closure of the set $C_0^\infty(\overline{\mathcal{K}} \setminus S)$ with respect to the norm

$$\|u\|_{V^l_{\beta,\delta}(\mathcal{K})} = \Big(\int\limits_{\mathcal{K}} \sum_{|\alpha| \le l} \rho^{2(\beta - l + |\alpha|)} \prod_{k=1}^{d} \Big(\frac{r_k}{\rho} \Big)^{2(\delta_k - l + |\alpha|)} |\partial_x^\alpha u|^2 dx \Big)^{\frac{1}{2}} < +\infty, \tag{3.2}$$

where $\rho = |x|$ is the distance of the point x from the origin 0, while r_k denotes the distance of the point x from the edge M_k. The closure of the set $C_0^\infty(\mathcal{K})$ with respect to the norm (3.2) is denoted by $\mathring{V}^l_{\beta,\delta}(\mathcal{K})$.

Obviously, from (3.2) we have the following imbedding

$$V^l_{\beta,\delta}(\mathcal{K}) \subset V^{l-1}_{\beta-1,\delta-1}(\mathcal{K}) \subset \cdots \subset V^0_{\beta-l,\delta-l}(\mathcal{K}).$$

We consider the Dirichlet problem for elliptic equations

$$\begin{cases} Lu = F \text{ on } \mathcal{K}, \\ \dfrac{\partial^k u}{\partial \nu^k}\Big|_{\Gamma_j} = 0, \quad j = 1, \cdots, d. \end{cases} \tag{3.3}$$

For the following lemma on the regularity of the solutions to elliptic boundary value problems in domains of polyhedral type, we refer to Corollary 4.1.10 and Theorem 4.1.11 of [22].

Lemma 3.2. *Let $u \in V^l_{\beta,\delta}(\mathcal{K})$ be a solution of the problem (3.3), where*

$$F \in V^{l-2m}_{\beta,\delta}(\mathcal{K}) \cap V^{l'-2m}_{\beta',\delta'}(\mathcal{K}), \quad l \ge m, \quad l' \ge m.$$

Suppose that the closed strip between the lines $\mathrm{Re}\lambda = l - \beta - \dfrac{3}{2}$ *and* $\mathrm{Re}\lambda = l' - \beta' - \dfrac{3}{2}$ *is free of eigenvalues of the pencil $\mathfrak{U}$ and that the components of δ and δ' satisfy the inequalities*

$$-\delta_+^{(k)} < \delta_k - l + m < \delta_-^{(k)}, \quad -\delta_+^{(k)} < \delta'_k - l' + m < \delta_-^{(k)}.$$

Then $u \in V^{l'}_{\beta',\delta'}(\mathcal{K})$ and

$$\|u\|^2_{V^{l'}_{\beta',\delta'}(\mathcal{K})} \le C \|F\|^2_{V^{l'-2m}_{\beta',\delta'}(\mathcal{K})},$$

where C is a constant independent of u and F.

From Lemma 3.1.3 and Lemma 3.1.6 in [22], for $\beta, \delta_k \in [-m, m],\ k = 1, 2, \ldots d$, we have following imbeddings

$$H^m(\mathcal{K}) \subset V^m_{0,0}(\mathcal{K}) \subset V^0_{\beta,\delta}(\mathcal{K}), \quad V^m_{0,0}(\mathcal{K}) \subset V^0_{-\beta,-\delta}(\mathcal{K}),$$

and

$$V^0_{\beta,\delta}(\mathcal{K}) \subset V^{-m}_{0,0}(\mathcal{K}),$$

where $V^{-m}_{0,0}(\mathcal{K})$ is the dual space of $V^m_{0,0}(\mathcal{K})$.

We denote by $H^{m,h}_{\beta,\delta}(\mathcal{K}_T)$ the weighted Sobolev space of functions u defined in $\mathcal{K}_T$ with the norm

$$\|u\|^2_{H^{m,h}_{\beta,\delta}(\mathcal{K}_T)} = \int\limits_{\mathcal{K}_T} \Big(\sum_{|\alpha|\le m} \rho^{2(\beta-m+|\alpha|)} \prod_{k=1}^{d} \Big(\frac{r_k}{\rho}\Big)^{2(\delta_k-m+|\alpha|)} |\partial_x^\alpha u|^2 + \sum_{j=1}^{h} |u_{t^j}|^2 \Big) dxdt.$$

Theorem 3.3. *Let l, h be nonnegative integers, $l \geqslant 2m$, and $\beta \in \mathbb{R}, \delta = (\delta_1, \dots, \delta_d) \in \mathbb{R}^d$, $\beta, \delta_k \in [-m, m]$, $k = 1, 2, \cdots, d$. Assume that the following conditions are satisfied*

(i) $f_{t^k} \in L_2(\mathcal{K}_T) \cap V^{l-2m}_{\beta,\delta}(\mathcal{K}_T), \quad k = 0, 1, \cdots, h+1,$

(ii) $f_{t^k}(x, 0) = 0, \quad k = 0, 1, \cdots, h-1.$

Additionally, suppose that the closed strip between the lines $\mathrm{Re}\lambda = m - 3/2$ *and* $\mathrm{Re}\lambda = l - \beta - 3/2$ *does not contain eigenvalues of the operator pencils $\mathfrak{U}(\lambda, t), t \in [0, T]$, and*

$$-\delta_+^{(k)} < \delta_k - l + m < \delta_-^{(k)}, \quad k = 1, \dots, d.$$

Let $u \in \mathring{H}^{m,1}(\mathcal{K}_T)$ be the generalized solution of problem (1.2)–(1.4). Then $u_{t^k} \in V^{l,0}_{\beta,\delta}(\mathcal{K}_T)$, $k = 0, 1, \cdots, h$, and

$$\sum_{k=0}^{h} \|u_{t^k}\|_{V^{l,0}_{\beta,\delta}(\mathcal{K}_T)} \le C \sum_{k=0}^{h} \|f_{t^k}\|_{V^{l-2m}_{\beta,\delta}(\mathcal{K}_T)} + \sum_{k=0}^{h+1} \|f_{t^k}\|_{L_2(\mathcal{K}_T)}, \tag{3.4}$$

where C is a constant independent of u and f.

Proof. The first, we prove theorem in the case of $l = 2m$ and for arbitrary $h \in \mathbb{N}$. Since $V^0_{\beta,\delta(\mathcal{K})} \subset V^{-m}_{0,0}(\mathcal{K})$ and $L_2(\mathcal{K}) \subset V^{-m}_{0,0}(\mathcal{K}), \beta, \delta_k \in [-m, m], k = 1, \dots, d$, we get from (i) that

$$f_{t^k} \in V^{-m}_{0,0}(\mathcal{K}) \cap V^0_{\beta,\delta}(\mathcal{K}), \quad k = 0, 1, \cdots, h+1. \tag{3.5}$$

Using the hypothesis (i), we have $f_{t^k} \in L_2(\mathcal{K}_T)$, $k = 0, 1, \cdots, h+1$. Thus, by Lemma 3.1, we obtain $u_{t^k} \in \mathring{H}^m(\mathcal{K}) \subset \mathring{V}^m_{0,0}(\mathcal{K})$, $k = 0, \cdots, h+1$. Furthermore, $\mathring{V}^m_{0,0}(\mathcal{K}) \subset V^0_{\beta,\delta}(\mathcal{K})$ for $\beta, \delta_k \in [-m, m], k = 1, \cdots, d$ so

$$u_{t^k} \in V^0_{\beta,\delta}(\mathcal{K}) = V^{-m}_{0,0}(\mathcal{K}) \cap V^0_{\beta,\delta}(\mathcal{K}). \tag{3.6}$$

We have from (3.5) and (3.6) that $f - u_t \in V^0_{\beta,\delta}(\mathcal{K})$, a.e. $t \in [0, T]$. Applying Lemma 3.2 for following problem

$$Lu = (-1)^m (f - u_t) \quad \text{in} \quad \mathcal{K}, \tag{3.7}$$

$$\frac{\partial^k u}{\partial \nu^k}\Big|_{\Gamma_j} = 0, \quad j = 1, \cdots, d, \tag{3.8}$$

(in the case of $l = m$, $\beta = 0$, $\delta_k = 0$, $l' = 2m$, $\beta' = \beta$, $\delta' = \delta$), we obtain $u(t) \in V^{2m}_{\beta,\delta}(\mathcal{K})$, a.e. $t \in [0, T]$, and

$$\|u(t)\|_{V^{2m}_{\beta,\delta}(\mathcal{K})} \le C\|f - u_t\|_{V^0_{\beta,\delta}(\mathcal{K})} \le C\|f\|_{V^0_{\beta,\delta}(\mathcal{K})} + C\|u_t\|_{V^0_{\beta,\delta}(\mathcal{K})}.$$

Integrating with respect to t from 0 to T and using Lemma 3.1, we obtain

$$\begin{aligned}\|u\|_{V^{2m,0}_{\beta,\delta}(\mathcal{K}_T)} &\le C\Big(\|f\|_{V^0_{\beta,\delta}(\mathcal{K}_T)} + \|u_t\|_{H^m(\mathcal{K}_T)}\Big) \le \\ &\le C\Big(\|f\|_{V^0_{\beta,\delta}(\mathcal{K}_T)} + \|f\|_{L_2(\mathcal{K}_T)} + \|f_t\|_{L_2(\mathcal{K}_T)}\Big),\end{aligned}$$

where C is a constant indefendent of f and u. Thus the assertions of the theorem hold for $h=0$ (in the case $l=2m$). Now, assume inductive they are true for $h-1$. Differentiating both sides of (3.7) and (3.8) h times with respect to t, we have

$$\begin{cases} Lu_{t^h} = \widehat{F} := (-1)^m\Big(f_{t^h} + u_{t^{h+1}} + \sum\limits_{k=0}^{h-1}\binom{h}{k}L_{t^{h-k}}u_{t^k}\Big) = 0 \quad \text{in } \mathcal{K}, \\ \dfrac{\partial^k u_{t^h}}{\partial\nu^k}\Big|_{\Gamma_j} = 0, \quad j=1,\cdots,d, \quad k=0,\cdots,m-1, \end{cases}$$

where $L_{t^{h-k}} = \sum\limits_{|p|,|q|=0}^{m} D^p(a_{pqt^{h-k}}D^q)$. Set $\widehat{u} = u_{t^h}$, we get

$$L\widehat{u} = \widehat{F} \qquad \text{in } \mathcal{K}, \tag{3.9}$$

$$\frac{\partial^k\widehat{u}}{\partial\nu^k}\Big|_{\Gamma_j} = 0, \quad j=1,\cdots,d,\ k=0,\cdots,m-1. \tag{3.10}$$

From the inductive assumptions, we see that

$$\begin{aligned}\Big\|\sum_{k=0}^{h-1}\binom{h}{k}L_{t^{h-k}}u_{t^k}\Big\|_{V^0_{\beta,\delta}(\mathcal{K}_T)} &\le C\sum_{k=0}^{h-1}\|u_{t^k}\|_{V^{2m}_{\beta,\delta}(\mathcal{K}_T)} \le \\ &\le C\Big(\sum_{k=0}^{h-1}\|f_{t^k}\|_{V^0_{\beta,\delta}(\mathcal{K}_T)} + \sum_{k=0}^{h}\|f\|_{L_2(\mathcal{K}_T)}\Big),\end{aligned}$$

and $u_{t^{h+1}} \in \overset{\circ}{H}{}^m(\mathcal{K}) \subset V^0_{\beta,\delta}(\mathcal{K})$. This together the hypothesis (i) imply that $\widehat{F} \in V^0_{\beta,\delta}(\mathcal{K})$, a.e. $t \in [0,T]$. Therefore $\widehat{F} \in L_2(\mathcal{K})$. Thus, we can use the same arguments as above to get from (3.9), (3.10) that $\widehat{u} = u_{t^h} \in V^{2m,0}_{\beta,\delta}(\mathcal{K}_T)$ and

$$\|u_{t^h}\|_{V^{2m,0}_{\beta,\delta}(\mathcal{K}_T)} \le C\Big(\sum_{k=0}^{h}\|f_{t^k}\|_{V^0_{\beta,\delta}(\mathcal{K}_T)} + \sum_{k=0}^{h+1}\|f_{t^k}\|_{L_2(\mathcal{K}_T)}\Big).$$

The assertions of theorem hold for the case of $l=2m$ and $h\in\mathbb{N}$.

Next, we proof the theorem by induction on l for any h. Suppose that the theorem is true $l-1$, for any h. We have

$$\begin{cases} Lu = (-1)^m(f-u_t) =: F, \\ \dfrac{\partial^k u}{\partial\nu^k}\Big|_{\Gamma_j} = 0, \quad j=1,\cdots,d, \quad k=0.\cdots,m-1. \end{cases} \tag{3.11}$$

From hypothesis $f \in V^{l-2m}_{\beta,\delta}(\mathcal{K})$, and by inductive assertions $u_t \in V^{l-1}_{\beta,\delta}(\mathcal{K}) \subset V^{l-2m}_{\beta,\delta}(\mathcal{K})$. According to Lemma 3.2, we obtain $u \in V^l_{\beta,\delta}(\mathcal{K})$. Differentiating both sides of (3.11) with respect to t, we get

$$\begin{cases} Lu_t = (-1)^m\big(f_t - u_{tt} - L_t u\big) =: F_1, \\ \dfrac{\partial^k u_t}{\partial \nu^k}\Big|_{\Gamma_j} = 0, \quad j = 1, \cdots, d, \quad k = 0, \cdots, m-1. \end{cases}$$

By the same arguments as above, we have $u_t \in V^l_{\beta,\delta}(\mathcal{K})$. Continuing above process, differentiating both sides of (3.11) i times with respect to t, we get

$$\begin{cases} Lu_{t^i} = F_i \\ \dfrac{\partial^k u_{t^i}}{\partial \nu^k}\Big|_{\Gamma_j} = 0, \quad j = 1, \cdots, d, \; k = 0, \cdots, m-1. \end{cases}$$

where $F_i := (-1)^m\Big(f_{t^i} - u_{t^{i+1}} - \sum\limits_{k=0}^{i-1} \binom{h}{k} L_{t^{i-k}} u_{t^k}\Big)$, $i \le h$. Notice that

$$f_{t^i} \in V^{l-2m}_{\beta,\delta}(\mathcal{K}), \quad u_{t^{i+1}} \in V^{l-1}_{\beta,\delta}(\mathcal{K}) \subset V^{l-2m}_{\beta,\delta}(\mathcal{K}),$$

and

$$L_{t^{i-k}} u_{t^k} \in V^{l-2m}_{\beta,\delta}(\mathcal{K}), \quad k \le i-1.$$

Therefore, $F_i \in V^{l-2m}_{\beta,\delta}(\mathcal{K})$. Applying Lemma 3.2 again, we obtain $u_{t^i} \in V^{l,0}_{\beta,\delta}(\mathcal{K}_T)$ and

$$\sum_{k=0}^{i} \|u_{t^k}\|_{V^{l,0}_{\beta,\delta}(\mathcal{K}_T)} \le \Big(\sum_{k=0}^{i} \|f_{t^k}\|_{V^{l-2m}_{\beta,\delta}(\mathcal{K}_T)} + \sum_{k=0}^{i+1} \|f_{t^k}\|_{L_2(\mathcal{K}_T)}\Big).$$

The proof is completed.

Remark: Let $r = \min\limits_{1\le k\le d} r_k$. Then there exist positive constants C_1, C_2 independent of x such that

$$C_1\rho(x)\prod_{k=1}^{d}\frac{r_k(x)}{\rho(x)} \le r \le C_2\rho(x)\prod_{k=1}^{d}\frac{r_k(x)}{\rho(x)}, \quad \text{for all} \quad x \in \mathcal{K}.$$

Thus, the norm in $V^l_\beta(\mathcal{K}) := V^l_{\beta,\beta}(\mathcal{K})$ equivalent following norm

$$\|u\|_{V^l_\beta(\mathcal{K})} = \Big(\sum_{|\alpha|\le l}\int_{\mathcal{K}} r^{2(\beta-l+|\alpha|)}|D^\alpha u|^2\,dx\Big)^{\frac{1}{2}}.$$

From Theorem 3.3, we get the following theorem.

Theorem 3.4. *Let l, h be nonnegative integers, $l \ge 2m, \beta \in \mathbb{R}, \beta \in [-m, m]$. Assume that the following conditions are satisfied*

1) $f_{t^k} \in V^{l-2m,0}_\beta(\mathcal{K}_T), \quad k = 0, \cdots, h+1,$

2) $f_{t^k}(x, 0) = 0, \quad k \le h.$

Additionally, suppose that the closed strip between the lines $\mathrm{Re}\lambda = m - 3/2$ *and* $\mathrm{Re}\lambda = l - \beta - 3/2$ *does not contain eigenvalues of the operator pencils* $\mathfrak{U}(\lambda, t), t \in [0, T]$*, and*

$$-\delta^{(k)}_k < \beta - l + m < \delta^{(k)}_-.$$

Let $u \in \overset{\circ}{H}{}^{m,1}(\mathcal{K}_T)$ be the generalized solution of problem (1.2)–(1.4). Then $u_{t^k} \in V_\beta^{l,0}(\mathcal{K}_T)$, $k = 0, \cdots, h$ and

$$\sum_{k=0}^{h} \|u_{t^k}\|_{V_\beta^{l,0}(\mathcal{K}_T)} \leq C \sum_{k=0}^{h} \|f_{t^k}\|_{V_\beta^{l-2m}(\mathcal{K}_T)} + \sum_{k=0}^{h+1} \|f_{t^k}\|_{L_2(\mathcal{K}_T)}. \tag{3.12}$$

In the case $m = 1$, we have similar results for problem (1.2)–(1.4) in a polyhedral domain, which is proven in [21].

4. An example. To illustrate the Theorem 3.3, in this section we consider as example the case of operator $L = \Delta$. For the following information concerning the eigenvalues of pencils $A_k(\lambda, t)$ and $\mathfrak{U}(\lambda, t)$ introduced in the previous section, we refer to [18, Chapter 2]. The eigenvalue of the operator pencil $A_k(\lambda)$ are

$$\lambda_j = j\pi/\theta_k, \ \ j = \pm 1, \pm 2, \ldots,$$

(see [18, section 2.1.1]). We see that $\delta_+^{(k)} = \delta_-^{(k)} = \pi/\theta_k$ are the greatest positive real numbers such that the strip

$$-\pi/\theta_k < \mathrm{Re}\lambda < \pi/\theta_k$$

is free of eigenvalues of the pencils $A_k(\lambda)$.

Let $\hat{\lambda}$ be the eigenvalues of the Laplace—Beltrami operator $-\delta$ (with the Dirichlet condition) on the subdomain Ω of the unit sphere (Ω is defined in the previous section). Then the eigenvalues of the pencils $\mathfrak{U}(\lambda)$ are given by

$$\Lambda_{\pm k} = -\frac{1}{2} \pm \sqrt{\hat{\lambda} + 1/4}.$$

It is well-known that the spectrum $-\delta$ is a countable set of positive eigenvalues (see [18, section 2.2.1]). Hence, the interval $[-1, 0]$ is free of eigenvalues of the pencils $\mathfrak{U}(\lambda)$. We denote the smallest positive eigenvalue of the $\mathfrak{U}(\lambda)$ by Λ^+. Then the interval $[-1-\Lambda_j^+, \Lambda_j^+]$ does not contain eigenvalues of the pencils $\mathfrak{U}(\lambda)$. Now, the conditions about the eigenvalues of pencils $A_k(\lambda)$ and $\mathfrak{U}(\lambda)$ in Theorem 3.3 can be written down simply as follows

$$-1 - \Lambda^+ < m - 3/2, \quad l - \beta - 3/2 < \Lambda^+,$$

and

$$|\delta_k + l - m| < \pi/\theta_k, \quad k = 1, \ldots, d.$$

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Сведения об авторах

Vu Trong Luong — Department of Mathematics, Taybac University, Sonla city, Sonla, Vietnam; vutrongluong@gmail.com

Do Van Loi — Department of Mathematics, Hongduc University, Thanhhoa, Vietnam; 37loilinh@gmail.com