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On a question concerning $D4$-modules

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An R-module M is called a *D4-module* if 'whenever M_1 and M_2 are direct summands of M with $M_1 + M_2 = M$ and $M_1 \cong M_2$, then $M_1 \cap M_2$ is a direct summand of M'. Let $M = \oplus_{i\in I} M_i$ be a direct sum of submodules M_i with $Hom(M_i, M_j) = 0$ for distinct $i, j \in I$. We show that M is a $D4$-module if and only if for each $i \in I$ the module M_i is a $D4$-module. This settles an open question concerning direct sums of $D4$-modules. Our approach is independent of the solution obtained by D'Este, Keskin Tütüncü and Tribak recently.

Keywords: SIP-modules, $D4$-modules.

1. Introduction. By a ring we mean an associative ring with an identity element; modules are unitary.

A module M is said to be a *SIP-module* (*SSP-module*) if the intersection (respectively, the sum) of two direct summands of M is a direct summand of M. Kaplansky observed that over a commutative principal ideal domain every free module is a SIP-module (see [1, Exercise 51(a), p. 49].) SIP-modules and SSP-modules have been extensively studied (see, for example, [2–4] and [5]).

For $1 \leq i \leq 4$, a module M is called a *Di-module* if it satisfies the condition Di noted below.

$D1$. For every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2$ is small in M_2.

$D2$. If $A \leq M$ such that M/A is isomorphic to a direct summand of M, then A is a direct summand of M.

$D3$. If M_1 and M_2 are direct summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_2$ is a direct summand of M.

$D4$. If M_1 and M_2 are direct summands of M with $M_1 + M_2 = M$ and $M_1 \cong M_2$, then $M_1 \cap M_2$ is a direct summand of M.

(For a detailed background of these notions, we refer to [6, Chapter 4] and to [7].)

A module M is also called a *lifting module* if it satisfies condition $D1$ (see [8] for detailed information regarding these modules). We recall the characterization "the ring R is semiperfect if and only if R is lifting as a right (or left) R-module" (see [9, Theorem 1.2.13]). Now let R be a commutative domain with zero Jacobson radical which is not a field, and hence is not semiperfect. Then, by the above results, ${}_RR$ is a projective module which is not a $D1$-module. We have, however, projective $\Longrightarrow$ quasi-projective $\Longrightarrow$

$D2$-module $\implies$ $D3$-module $\implies$ $D4$-module (see [6, Proposition 4.38 and Lemma 4.6]). Note that for all proper subgroups N of the (indecomposable) Prüfer p-group $M = Z_{p^\infty}$, the group M/N is isomorphic to M. Hence it is $D3$ (as a $\mathbb{Z}$-module) but not $D2$. In fact, there are rings over which every cyclic module is $D3$ but not all cyclic modules are $D2$ (see [10, Example 6.4]).

There is no known example of a module which is $D4$ but not $D3$ [11] (see also [12, p. 2]).

Let A and B be right R-modules. A homomorphism $f \in Hom_R(A, B)$ is said to be (*von Neumann*) *regular* (briefly, *regular*) if for some homomorphism $g \in Hom_R(B, A)$, we have the relation $f = fgf$. It is well-known that a homomorphism $f \in Hom_R(A, B)$ is regular if and only if $Ker(f)$ is a direct summand in A and $Im(f)$ is a direct summand in B.

Recall that a module M is called a *Rickart module* if the kernel of any endomorphism $f \in End_R(M)$ is a direct summand in M. It follows from [13, Proposition 2.16] that every Rickart module is a SIP-module. A module M is called a dual Rickart module if the image of any endomorphism $f \in End_R(M)$ is a direct summand in M. It follows from [14, Proposition 2.11] that every dual Rickart module is a SSP-module.

2. Results. We begin with the recall of some results from [15].

Lemma 1 [15, Lemma 2.1]. *Let M be a right R-module, $f, g \in End_R(M)$ be regular homomorphisms, and let*

$$M = Ker(f) \oplus A = Im(f) \oplus B, M = Ker(g) \oplus A' = Im(g) \oplus B'.$$

Then the following assertions hold:

(a) $Im(fg) = f(A \cap (Im(g) + Ker(f)))$;

(b) $Ker(fg) = (g|_{A'})^{-1}(Im(g) \cap Ker(f)) + Ker(g)$.

Lemma 2 [15, Lemma 2.2]. *Let M be a right R-module, π be the projection onto the first direct summand with respect to the decomposition $M = A_1 \oplus A_2$, and let π' be the projection onto the first direct summand with respect to the decomposition $M = B_1 \oplus B_2$. Then the following assertions hold:*

(a) $Im(\pi'\pi) = (A_1 + B_2) \cap B_1$;

(b) $Ker(\pi'\pi) = (A_1 \cap B_2) + A_2$.

Proposition 1 [15, Theorem 2.3]. *For a right R-module M, the following conditions are equivalent.*

1. *M is a SSP-module.*

2. *For any two regular homomorphisms $f, g \in End_R(M)$, the module $Im(fg)$ is a direct summand of the module M.*

Proposition 2 [15, Theorem 2.4]. *For a right R-module M, the following conditions are equivalent.*

1. *M is a SIP-module.*

2. *For any two regular homomorphisms* $f, g \in End_R(M)$*, the module* $Ker(fg)$ *is a direct summand of the module* M*.*

Next we note examples of finite abelian groups which are not $D4$.

Example. Consider $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ as a $\mathbb{Z}$-module. Then $A = (\bar{1}, \bar{3})\mathbb{Z}$ and $B = (\bar{0}, \bar{3})\mathbb{Z}$ are isomorphic direct summands of M. However, $A \cap B$ is not a direct summand of M. In fact, for any prime p, consider $M = \mathbb{Z}/p^m\mathbb{Z} \oplus \mathbb{Z}/p^n\mathbb{Z}$ with $n > m$ as a $\mathbb{Z}$-module, then M is not a $D4$-module, since there is an epimorphism $\mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^m\mathbb{Z}$ whose kernel is not a direct summand of $\mathbb{Z}/p^n\mathbb{Z}$.

The following theorem is an analogue of [15, Theorem 3.3].

Theorem 1. *For a right* R*-module* M*, consider the following statements.*

1. M *is a D3-module.*

2. *For any two regular endomorphisms* $f, g \in End_R(M)$*, if* $Im(fg)$ *is a direct summand of the module* M*, then the module* $Ker(fg)$ *is a direct summand of the module* M*.*

3. *For any two regular endomorphisms* $f, g \in End_R(M)$ *satisfying the following:*

(i) $Im(fg)$ *is a direct summand of the module* M*,*

(ii) $Ker(f) \cong Im(g)$*,*

then the module $Ker(fg)$ *is a direct summand of the module* M*.*

4. M *is a D4-module.*

5. *For any two regular endomorphisms* $f, g \in End_R(M)$ *satisfying the following:*

(i) $Im(fg)$ *is a direct summand of the module* M*,*

(ii) $N + Ker(f) \cong Im(g)$ *for any direct summand* N *of* M *such that* $N \cap Ker(f) = 0$*,*

then the module $Ker(fg)$ *is a direct summand of the module* M*.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

PROOF. (1) $\Leftrightarrow$ (2) follows from [15, Theorem 3.3].
(2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (4). Let $M = A \oplus A' = B \oplus B'$, where $A + B = M$ and $A \cong B$. Consider the natural projections $\pi_1 : A \oplus A' \longrightarrow A$ and $\pi_2 : B \oplus B' \longrightarrow B'$. Then by Lemma 2(a), $Im(\pi_2\pi_1) = B'$ is a direct summand of M. Therefore by assumption and Lemma 2(b), $Ker(\pi_2\pi_1) = (A \cap B) \oplus A'$ is a direct summand of M. This shows that $A \cap B$ is a direct summand of M, as required.
(4) $\Rightarrow$ (5). Let

$$M = Ker(f) \oplus A = Im(f) \oplus B = Ker(g) \oplus A' = Im(g) \oplus B'.$$

By Lemma 1(a), since $f|_A$ is an isomorphism $(Im(g) + Ker(f)) \cap A$ is a direct summand of M. Therefore, $A = N \oplus (Im(g) + Ker(f)) \cap A$, for some $N \leq A$. Since $(N + Ker(f)) + Im(g) = M$, $N + Ker(f) \cong Im(g)$ and M is a $D4$-module, we have $(N + Ker(f)) \cap$

$Im(g) = (Ker(f) \cap Im(g))$ is a direct summand of M. Since $g|_{A'} : A' \longrightarrow Im(g)$ is an isomorphism, we have $(g|_{A'})^{-1}(Im(g) \cap Ker(f))$ is a direct summand of M. Hence by Lemma 1(b), $Ker(fg)$ is a direct summand of M. □

Recall that a module M is called a *C3-module* if A and B are direct summands in M with $A \cap B = 0$, then $A \oplus B$ is a direct summand in M.

Following Ding et al. [16, Theorem 2.2(5)], a module M is called a *C4-module* if A and B are isomorphic direct summands in M with $A \cap B = 0$, then $A \oplus B$ is a direct summand in M. Clearly $C3$-modules are $C4$-modules. However, there are examples of $C4$-modules which are not $C3$.

The following theorem is an analogue of [15, Theorem 3.1].

Theorem 2. *For a right R-module M, consider the following statements.*

1. *M is C3-module.*

2. *For any two regular endomorphisms $f, g \in End_R(M)$, if $Ker(fg)$ is a direct summand of the module M, then the module $Im(fg)$ is a direct summand of the module M.*

3. *For any two regular endomorphisms $f, g \in End_R(M)$ satisfying the following:*

(i) *$Ker(fg)$ is a direct summand of the module M,*

(ii) *$Ker(f) \cong Im(g)$,*

then the module $Im(fg)$ is a direct summand of the module M.

4. *M is a C4-module.*

5. *For any two regular endomorphisms $f, g \in End_R(M)$ satisfying the following:*

(i) *$Ker(fg)$ is a direct summand of the module M,*

(ii) *$N \cong Im(g)$ for any direct summand N of $Ker(f)$,*

then the module $Im(fg)$ is a direct summand of the module M.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

Proof. (1) $\Leftrightarrow$ (2) follows from [15, Theorem 3.1].

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). Let $M = A \oplus A' = B \oplus B'$, where $A \cap B = 0$ and $A \cong B$. Consider the natural projections $\pi_1 : A \oplus A' \longrightarrow A$ and $\pi_2 : B \oplus B' \longrightarrow B'$. Then by Lemma 2(b), $Ker(\pi_2\pi_1) = A'$ is a direct summand of M. Therefore by assumption and Lemma 2(a), $Im(\pi_2\pi_1) = (A + B) \cap B'$ is a direct summand of M. Since $A + B = B \oplus (A + B) \cap B'$, $A + B$ is a direct summand of M, as required.

(4) $\Rightarrow$ (5). Let

$$M = Ker(f) \oplus A = Im(f) \oplus B = Ker(g) \oplus A' = Im(g) \oplus B'.$$

By Lemma 1(b), $(g|_{A'})^{-1}(Im(g) \cap Kerf)$ is a direct summand of A'. Since $g|_{A'} : A' \longrightarrow Im(g)$ is an isomorphism and $Im(g)$ is a direct summand of the module M, we have that $Im(g) \cap Ker(f)$ is a direct summand of the module M. Therefore, $Ker(f) = N \oplus (Im(g) \cap Ker(f))$, for some $N \leq M$. Since $N \cap Im(g) = 0$, $N \cong Im(g)$ and M is a

$C4$-module, we have $N \oplus Im(g)$ is a direct summand of M. Since $Ker(f) \leq Im(g) \oplus N$, we have that

$$Im(g) \oplus N = Ker(f) \oplus (Im(g) + N) \cap A = Ker(f) \oplus (Im(g) + Ker(f)) \cap A.$$

Therefore, $(Im(g) + Ker(f)) \cap A$ is a direct summand of M. Hence by Lemma 1(a), $Im(fg)$ is a direct summand of M. □

We can now prove the following result which has already appeared in [17, Proposition 5.7 and Corollary 2.9]. The proof has been outlined by us for the sake of completeness.

Proposition 3. *For a right R-module M, the following conditions are equivalent.*

1. *M is a D4-module and a SSP-module.*
2. *M is a C3-module and a SIP-module.*
3. *M is a C4-module and a SIP-module.*
4. *M is a D3-module and a SSP-module.*
5. *M is an SSP-module and a SIP-module.*

Proof. (1) $\Longrightarrow$ (2). Let M be a SSP-module. It is clear that M is a $C3$-module. To see that M is a SIP-module, we shall use Proposition 2. Let $f, g \in End_R(M)$ be two regular endomorphisms such that

$$M = Ker(f) \oplus A = Im(f) \oplus B = Ker(g) \oplus A' = Im(g) \oplus B'.$$

We need to show that $Ker(fg)$ is a direct summand of M. By Lemma 1(b), enough to show that $Im(g) \cap Ker(f)$ is a direct sumand of M. To this end we shall follow the proof of [3, Proposition 1.4]. Let $\pi_1 : Im(g) \oplus B \longrightarrow Im(g)$ and $\pi_2 : Ker(f) \oplus A \longrightarrow Ker(f)$ be the natural projections. Define $\theta = ((\pi_1 - 1) \circ \pi_2)|_{Im(g)} : Im(g) \longrightarrow B'$. Then by [2, Proposition 1.4], $Im(\theta)$ is a direct summand of B'. Hence M being a D4-module (use [7, Theorem 2.2]), we have $Ker(\theta) = (Im(g) \cap Ker(f)) \oplus (Im(g) \cap A)$ is a direct summand of $Im(g)$. Thus $Im(g) \cap Ker(f)$ is a direct sumand of M, as desired.

(2) $\Longrightarrow$ (3) is clear.

(3) $\Longrightarrow$ (4). Let M be a SIP-module. It is clear that M is a $D3$-module. To see that M is a SSP-module, we shall use Proposition 1. Let $f, g \in End_R(M)$ be two regular endomorphisms such that

$$M = Ker(f) \oplus A = Im(f) \oplus B = Ker(g) \oplus A' = Im(g) \oplus B'.$$

We need to show that $Im(fg)$ is a direct summand of M. By Lemma 1(a), enough to show that $Im(g) + Ker(f)$ is a direct sumand of M. To this end we shall follow the proof of [5, Theorem 8]. Let $\pi_1 : Ker(f) \oplus A \longrightarrow Ker(f)$ and $\pi_2 : Im(g) \oplus B' \longrightarrow B'$ be the natural projections. Define $\phi = (\pi_2 \circ \pi_1)|_{Im(g)} : Im(g) \longrightarrow B'$. Then by [3, Proposition 1.4], $Ker(\phi)$ is a direct summand of B'. Hence M being a $C4$-module (use [16, Theorem 2.2]), we have $Im(\phi) = [Im(g) + Ker(f)] \cap [Im(g) + A] \cap B'$ is a direct summand of $Im(g)$. So we can write $M = Im(\phi) \oplus X$ for some $X \leq M$. Hence $B' = Im(\phi) \oplus (B' \cap X)$. Then we have $M = [Im(g) + Ker(f)] \oplus [(Im(g) + A) \cap (B' \cap X)]$, as required.

(4) $\Longrightarrow$ (5) follows from Proposition 2 and Theorem 1.

(5) $\implies$ (1) is clear. □

The following result extends [15, Lemma 4.2(2)].

Proposition 4. *Let M be a dual Rickart module. If M is a D4-module, then the product of any two regular elements in the ring $End_R(M)$ is a regular element.*

PROOF. It follows from the hypothesis and Proposition 3 that M is a SSP-module and a SIP-module. Hence the result follows from [15, Theorem 2.7]. □

The following theorem was proved in [17].

Theorem 3 [17, Theorem 5.6]. *Let $M = \oplus_{i\in I} M_i$ be a direct sum of submodules M_i. If $N = \oplus_{i\in I}(N \cap M_i)$ for every submodule N of M, then M is a D4-module if and only if for each $i \in I$, M_i is a D4-module.*

In [17], immediately after Theorem 3 the following question was asked.

Question (see [17, Question, p. 4494]). It is known that if $N = \oplus_{i\in I}(N \cap M_i)$ for every submodule N of M, then $Hom(M_i, M_j) = 0$ for every $i \neq j$ in I, so it is natural to ask if [17, Theorem 5.6] (that is the theorem above) remains true if one assumes that $Hom(M_i, M_j) = 0$ for every $i \neq j$ in I.

In the next proposition we show that Question above has a positive answer.

Proposition 5. *Let $M = \oplus_{i\in\mathbb{N}} M_i$ be a direct sum of submodules M_i in which $Hom(M_i, M_j) = 0$ for every $i \neq j$. Then the following assertions hold:*

(i) *if M is a D4-module, then for each $i \in I$, M_i is a D4-module,*

(ii) *if each M_i is a D4-module, then M is a D4-module.*

PROOF. (i). Since a direct summand of a $D4$-module is a $D4$-module (see [7, Proposition 2.11]), for every $i \in \mathbb{N}$, M_i is a $D4$-module if M is a $D4$-module.

(ii). By hypothesis and [18, the paragraph before Corollary 16.5], we have

$$End_R(M) \cong \begin{pmatrix} End_R(M_1) & 0 & 0 & \cdot\cdot & \cdot\cdot \\ 0 & End_R(M_2) & 0 & \cdot\cdot & \cdot\cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdot\cdot & End_R(M_n) & \cdot\cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{\mathbb{N}\times\mathbb{N}}.$$

Take two regular elements f, g in $End_R(M)$ such that $Im(fg)$ is a direct summand of M and $Ker(f) \cong Im(g)$. Then $f = (f_i)_{i\in\mathbb{N}}$ and $g = (g_i)_{i\in\mathbb{N}}$ for some regular elements f_i and g_i in $End_R(M_i)$ such that $Im(f_i g_i)$ is a direct summand of M_i and $[X_i + Ker(f_i)] \cong Im(g_i)$ for any direct summand X_i of M_i such that $X_i \cap Ker(f_i) = 0$ for all $i \in \mathbb{N}$. But then each M_i is a $D4$-module. Therefore by Theorem 1, $Ker(f_i g_i)$ is a direct summand of M_i for all $i \in \mathbb{N}$. Hence $Ker(fg)$ is a direct summand of M, as required. □

Remark. Let $\{p_i\}_{i\in\mathbb{N}}$ be an infinite set of prime numbers and let p be a prime different from any of them. Then we have the following examples of $D4$-modules:

(i) $M = \mathbb{Z}_{p^\infty} \oplus (\oplus_{i\in\mathbb{N}}\ \mathbb{Z}/p_i\mathbb{Z})$ as a $\mathbb{Z}$-module, where $\mathbb{Z}_{p^\infty}$ is the Prüfer p-group;

(ii) $M = \mathbb{Q} \oplus (\oplus_{i\in\mathbb{N}}\ \mathbb{Z}/p_i\mathbb{Z})$ as a $\mathbb{Z}$-module.

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К вопросу о $D4$-модулях

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R-модуль M называется $D4$-*модулем*, если всякий раз, когда M_1 и M_2 являются прямыми слагаемыми M с $M_1 + M_2 = M$ и $M_1 \cong M_2$, то $M_1 \backslash M_2$ является прямым слагаемым M. Пусть $M = \bigoplus_{i \in I} M_i$ — прямая сумма подмодулей M_i с $Hom(M_i; M_j) = 0$ для различных $i, j \in I$. Показано, что M является $D4$-модулем тогда и только тогда, когда для каждого $i \in I$ модуль M_i является $D4$-модулем. Это решает открытый вопрос о прямых суммах $D4$-модулей. Наш подход не зависит от решения, полученного недавно Д'Эсте, Кескином Тютюнджу и Трибаком.

Ключевые слова: SIP-модули, $D4$-модули.

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