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L_p-inequalities for the polar derivative of a lacunary-type polynomial

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In this paper, we extend an inequality concerning the polar derivative of a polynomial in L_p-norm to the class of lacunary polynomials and thereby obtain a bound that depends on some of the coefficients of the polynomial as well.

Keywords: L_p-inequalities, polar derivative, polynomials.

1. Zygmund type inequalities for polynomials. Let $f(x)$ be a real polynomial of degree at most n then according to a well-known classical result in approximation theory due to A. Markov [1],

$$\max_{-1\leq x\leq 1}|f'(x)| \leq n^2 \max_{-1\leq x\leq 1}|f(x)|.$$

The above inequality is best possible because for Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$, $\max_{-1\leq x\leq 1}|T_n(x)| = 1$ and $|T_n'(\pm 1)| = n^2$. This inequality has been generalized in several ways, in particular, S. Bernstein (for details see [2] or [3]) obtained its extension to complex polynomials. Let $\mathcal{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most n. According to Bernstein's inequality, if $P \in \mathcal{P}_n$ then

$$\|P'\|_\infty \leq n\,\|P\|_\infty\,, \quad \text{where} \quad \|P\|_\infty := \max_{|z|=1}|P(z)|\,.$$

Define the standard Hardy space H^p norm for $P \in \mathcal{P}_n$ by

$$\|P\|_p := \left(\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right)^{1/p}, \; 0 < p < \infty.$$

It is well known that the supremum norm of the space H^∞ satisfies

$$\|P\|_\infty = \lim_{p\to\infty}\|P\|_p\,.$$

The other limiting case, also known as *Mahler measure* of a polynomial $P(z)$, is

$$\|P\|_0 := \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log\left|P(e^{i\theta})\right| d\theta\right).$$

If $P \in \mathcal{P}_n$, then

$$\|P'\|_p \leq n\|P\|_p. \tag{1}$$

Inequality (1) is due to Zygmund [4]. Zygmund obtained this inequality as an analogue to Bernstein's inequality. Arestov [5] showed that the inequality (1) remains valid for $0 < p < 1$ as well. Equality in (1) holds for $P(z) = \alpha z^n, \alpha \neq 0$.

For the class of polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < 1$, inequality (1) can be sharpened. In fact, if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < 1$, then

$$\|P'\|_p \leq \frac{n}{\|1+z\|_p} \|P\|_p, \; p \geq 1. \tag{2}$$

Inequality (2) was found out by De Bruijn [6]. Rahman and Schmeisser [7] proved the inequality (2) remains true for $0 < p < 1$ as well.

The estimates is sharp and equality in (2) holds for $P(z) = az^n + b, |a| = |b| \neq 0$.

Govil and Rahman [8] generalized inequality (2) and proved that if $P \in \mathcal{P}_n$ does not vanish in $|z| < k$ where $k \geq 1$, then

$$\|P'\|_p \leq \frac{n}{\|k+z\|_p} \|P\|_p, \; p \geq 1. \tag{3}$$

As a refinement of inequality (3), it was shown by Rather [9] that if $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ for $|z| < k, k \geq 1$, then

$$\|P'\|_p \leq \frac{n}{\|\delta_1+z\|_p} \|P\|_p, \; p > 0. \tag{4}$$

where δ_1 is defined by

$$\delta_1 = k^2 \frac{\frac{1}{n}\frac{|a_1|}{|a_0|} + 1}{\frac{1}{n}\frac{|a_1|}{|a_0|}k^2 + 1}. \tag{5}$$

2. Extension of Zygmund type inequalities to polar derivatives. By Gauss—Lucas theorem (see [10]), if all the zeros of a polynomial $P \in \mathcal{P}_n$ of degree n lie in a half plane then its critical points are also contained therein. Since we may map a half plane onto a closed disk through a bilinear transformation $z = \phi(w) = \frac{aw+b}{aw+d}$, $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Let $g(w) = (cw+d)^n P\left(\frac{aw+b}{cw+d}\right)$ be transformation of $P(z)$ under ϕ, then if ξ is a critical point of g, then $\phi(\xi)$ is either ∞ or a zero of the polynomial $nP(z) + \left(\frac{a}{c} - z\right) P'(z)$. This property guides us to the polynomial

$$D_\alpha[P](z) := nP(z) + (\alpha - z)P'(z),$$

called the polar derivative of P with respect to a complex number α (for details see [10, p. 44]). Note that $D_\alpha[P](z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha[P](z)}{\alpha} = P'(z)$$

uniformly with respect z for $|z| \leq R, R > 0$.

Aziz and Rather [11] extended inequality (2) to the polar derivative of a polynomial and proved that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, and $p \geq 1$,

$$\|D_\alpha[P]\|_p \leq n\left(\frac{|\alpha|+1}{\|1+z\|_p}\right)\|P\|_p . \tag{6}$$

Aziz et al. [12] also obtained an analogue of inequality (3) to the polar derivative and proved that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $p \geq 1$,

$$\|D_\alpha[P]\|_p \leq n\left(\frac{|\alpha|+k}{\|k+z\|_p}\right)\|P\|_p . \tag{7}$$

Later N. A. Rather [13, 14] showed that inequalities (6) and (7) remain valid for $0 < p < 1$ as well.

Recently, Rather et al. [15] extended (4) to the polar derivative which among other things also include a refinement of (7) and proved if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $0 \leq p < \infty$,

$$\|D_\alpha P\|_p \leq n\left(\frac{|\alpha|+\delta_1}{\|\delta_1+z\|_p}\right)\|P\|_p , \tag{8}$$

where δ_1 is given by (5).

Let $\mathcal{P}_{n,\mu} \subset \mathcal{P}_n$ be a class of lacunary type polynomials $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, where $1 \leq \mu \leq n$.

As a generalization of inequality (8), they [15] also proved that if $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $0 \leq p < \infty$,

$$\|D_\alpha[P]\|_p \leq n\left(\frac{|\alpha|+\delta_\mu}{\|\delta_\mu+z\|_p}\right)\|P\|_p , \tag{9}$$

where

$$\delta_\mu = k^{\mu+1}\left\{\frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|}k^{\mu-1}+1}{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|}k^{\mu+1}+1}\right\} \qquad (\geq k^\mu).$$

3. Main results. Our main result is a compact generalization of all the above results for the class of polynomials not vanishing in $|z| < k$, $k \geq 1$. Here, we present our main result:

Theorem. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, $0 \leq p < \infty$ and $0 \leq t \leq 1$,*

$$\left\| |D_\alpha[P]| + nmt\left(\frac{|\alpha|-1}{1+\delta_{\mu,t}}\right)\right\|_p \leq n\left(\frac{|\alpha|+\delta_{\mu,t}}{\|z+\delta_{\mu,t}\|_p}\right)\|P\|_p \tag{10}$$

where $m = \min_{|z|=k}|P(z)|$ and

$$\delta_{\mu,t} = k^{\mu+1}\left\{\frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-tm}k^{\mu-1}+1}{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}+1}\right\}. \tag{11}$$

For $t = 0$, (10) reduces to (9). If in above theorem, we let $p \to \infty$, we obtain the following Corollary.

Corollary 3.1. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $0 \leq t \leq 1$,*

$$\|D_\alpha[P]\|_\infty \leq \frac{n}{1+\delta_{\mu,t}} \{(|\alpha| + \delta_{\mu,t})\|P\|_\infty - (|\alpha| - 1)tm\}$$

where $\delta_{\mu,t}$ is given by (11).

If we divide both sides of inequality (10) by $|\alpha|$ and let $|\alpha| \to \infty$, we obtain the following refinement of inequality (4).

Corollary 3.2. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $0 \leq p < \infty$ and $0 \leq t \leq 1$,*

$$\left\| |P'| + \frac{nmt}{1+\delta_{\mu,t}} \right\|_p \leq \frac{n}{\|z + \delta_{\mu,t}\|_p} \|P\|_p \tag{12}$$

where $m = \min_{|z|=k} |P(z)|$. The result is best possible as shown by the polynomial $P(z) = (z^\mu + k^\mu)^{n/\mu}$.

Inequality (12) also includes a refinement of (3). By taking $k = 1$ and $\mu = 1$ in (12), the following improvement of inequality (2), which holds uniforms for $0 \leq t \leq 1$, follows immediately.

Corollary 3.3. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$ then for $0 \leq p < \infty$ and $0 \leq t \leq 1$,*

$$\left\| |P'| + \frac{nmt}{2} \right\|_p \leq \frac{n}{\|1 + z\|_p} \|P\|_p \tag{13}$$

where $m = \min_{|z|=1} |P(z)|$. The result is sharp and equality in (13) *holds for $P(z) = z^n + 1$.*

4. Lemmas. For the proof of above theorem, we need the following lemmas. The first lemma is due to [16].

Lemma 4.1. *If $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in $|z| < k$ where $k \geq 1$, then for $0 \leq t \leq 1$,*

$$\delta_{\mu,t}|P'(z)| \leq |Q'(z)| - tmn \qquad for \quad |z| = 1$$

and $\delta_{\mu,t} \geq k^\mu \geq 1$ where $\delta_{\mu,t}$ is given by (11), *$Q(z) = z^n\overline{P(1/\overline{z})}$ and $m = \min_{|z|=k}|P(z)|$.*

Lemma 4.2. *If A, B and C are non-negative real numbers such that $B + C \leq A$, then for every real number β,*

$$|(A - C) + e^{i\beta}(B + C)| \leq |A + e^{i\beta}B|.$$

The above lemma is due to Aziz and Rather [17] and the next lemma is due to [15].

Lemma 4.3. *If a, b are any two positive real numbers such that $a \geq bc$ where $c \geq 1$, then for any $x \geq 1, p > 0$ and $0 \leq \beta < 2\pi$,*

$$(a+bx)^p \int_0^{2\pi} \left|c+e^{i\beta}\right|^p d\beta \leq (c+x)^p \int_0^{2\pi} \left|a+be^{i\beta}\right|^p d\beta.$$

We also need the following lemma due to Aziz and Rather [18].

Lemma 4.4. *If $P \in \mathcal{P}_n$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then for every $p > 0$ and β real, $0 \leq \beta < 2\pi$,*

$$\int_0^{2\pi}\int_0^{2\pi} \left|P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right|^p d\theta d\beta \leq 2\pi n^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta.$$

5. Proof of Theorem. By hypothesis $P \in \mathcal{P}_n$ does not vanish in $|z| < k$ where $k \geq 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, therefore, by Lemma 4.1, we have for $|z| = 1$,

$$\delta_{\mu,t}\left|P'(z)\right| \leq |Q'(z)| - tmn = |Q'(z)| - tmn\left(\frac{1+\delta_{\mu,t}}{1+\delta_{\mu,t}}\right).$$

Equivalently,

$$\delta_{\mu,t}\left(|P'(z)| + \frac{mnt}{1+\delta_{\mu,t}}\right) \leq |Q'(z)| - \frac{mnt}{1+\delta_{\mu,t}} \qquad \text{for} \quad |z| = 1. \tag{14}$$

Setting $A = \left|Q'(e^{i\theta})\right|$, $B = |P'(e^{i\theta})|$ and $C = \dfrac{mnt}{1+\delta_{\mu,t}}$ in Lemma 4.2 and noting by (14) that $B + C \leq \delta_{\mu,t}(B+C) \leq A - C \leq A$ since $\delta_{\mu,t} \geq 1$. Therefore, by Lemma 4.2 for each real β, we get

$$\left|\left(|Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}}\right) + e^{i\beta}\left(|P'(e^{i\theta})| + \frac{mnt}{1+\delta_{\mu,t}}\right)\right| \leq \left||Q'(e^{i\theta})| + e^{i\beta}|P'(e^{i\theta})|\right|.$$

This implies for each $p > 0$,

$$\int_0^{2\pi} \left|F(\theta) + e^{i\beta}G(\theta)\right|^p d\theta \leq \int_0^{2\pi} \left||Q'(e^{i\theta})| + e^{i\beta}|P'(e^{i\theta})|\right|^p d\theta \tag{15}$$

where

$$F(\theta) = |Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}} \ \text{ and } \ G(\theta) = |P'(e^{i\theta})| + \frac{mnt}{1+\delta_{\mu,t}}. \tag{16}$$

Integrating (15) both sides with respect to β from 0 to 2π and using properties of definite integrals, we get

$$\int_0^{2\pi}\int_0^{2\pi} \left|F(\theta) + e^{i\beta}G(\theta)\right|^p d\theta d\beta \leq \int_0^{2\pi}\int_0^{2\pi} \left||Q'(e^{i\theta})| + e^{i\beta}|P'(e^{i\theta})|\right|^p d\theta d\beta =$$
$$= \int_0^{2\pi}\int_0^{2\pi} \left|P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right|^p d\theta d\beta.$$

By using Lemma 4.4 this implies,

$$\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\beta}G(\theta)\right|^p d\theta d\beta \le 2\pi n^p \int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta. \tag{17}$$

Since $\delta_{\mu,t}\ge 1$, we have

$$|Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}}\delta_{\mu,t} \le |Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}}.$$

On adding $t\left(|P'(e^{i\theta})|+\dfrac{mnt}{1+\delta_{\mu,t}}\right)$ on both sides, where $0\le t\le 1$, we get

$$|Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}}\delta_{\mu,t} + t\left(|P'(e^{i\theta})|+\frac{mnt}{1+\delta_{\mu,t}}\right) \le$$
$$\le |Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}} + t\left(|P'(e^{i\theta})|+\frac{mnt}{1+\delta_{\mu,t}}\right).$$

This further implies for each $p>0$,

$$\int_0^{2\pi}\left\{|Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}}\delta_{\mu,t} + t\left(|P'(e^{i\theta})|+\frac{mnt}{1+\delta_{\mu,t}}\right)\right\}^p d\theta \le$$
$$\le \int_0^{2\pi}\left\{|Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}} + t\left(|P'(e^{i\theta})|+\frac{mnt}{1+\delta_{\mu,t}}\right)\right\}^p d\theta.$$

Now for $\alpha\in\mathbb{C}$ with $|\alpha|\ge 1$ and $0\le t\le 1$, we have

$$|D_\alpha[P](e^{i\theta})| + nmt\left(\frac{|\alpha|-\delta_{\mu,t}}{1+\delta_{\mu,t}}\right) \le |\alpha||P'(e^{i\theta})| + |Q'(e^{i\theta})| + nmt\left(\frac{|\alpha|-\delta_{\mu,t}}{1+\delta_{\mu,t}}\right) =$$
$$= |\alpha|\left(|P'(e^{i\theta})|+\frac{mnt}{1+\delta_{\mu,t}}\right) + \left(|Q'(e^{i\theta})|-\frac{mnt}{1+\delta_{\mu,t}}\right).$$

By integrating both sides with respect to θ from 0 to 2π, for each $p>0$, we get

$$\int_0^{2\pi}\left\{|D_\alpha[P](e^{i\theta})| + nmt\left(\frac{|\alpha|-\delta_{\mu,t}}{1+\delta_{\mu,t}}\right)\right\}^p d\theta \le$$
$$\le \int_0^{2\pi}\left\{|\alpha|\left(|P'(e^{i\theta})|+\frac{mnt}{1+\delta_{\mu,t}}\right) + \left(|Q'(e^{i\theta})|-\frac{mnt}{1+\delta_{\mu,t}}\right)\right\}^p d\theta.$$

Multiply both sides by $\int_0^{2\pi}|\delta_{\mu,t}+e^{i\beta}|^p d\beta$, we obtain

$$\int_0^{2\pi}|\delta_{\mu,t}+e^{i\beta}|^p d\beta \int_0^{2\pi}\left\{|D_\alpha[P](e^{i\theta})| + nmt\left(\frac{|\alpha|-\delta_{\mu,t}}{1+\delta_{\mu,t}}\right)\right\}^p d\theta \le$$
$$\le \int_0^{2\pi}\left\{|\alpha|\left(|P'(e^{i\theta})|+\frac{mnt}{1+\delta_{\mu,t}}\right) + \left(|Q'(e^{i\theta})|-\frac{mnt}{1+\delta_{\mu,t}}\right)\right\}^p d\theta \times \int_0^{2\pi}|\delta_{\mu,t}+e^{i\beta}|^p d\beta. \tag{18}$$

Further, since $\delta_{\mu,t} \geq 1$, $1 \leq \mu \leq n$, by Lemma 4.3 with $a = \left|Q'(e^{i\theta})\right| - \frac{mnt}{1+\delta_{\mu,t}}$, $b = \left|P'(e^{i\theta})\right| + \frac{mnt}{1+\delta_{\mu,t}}$, $c = \delta_{\mu,t}$ and $x = |\alpha|$, we get for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$,

$$\left\{\left(|Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}}\right) + |\alpha|\left(|P'(e^{i\theta})| + \frac{mnt}{1+\delta_{\mu,t}}\right)\right\}^p \int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}|^p d\beta \leq$$
$$\leq (|\alpha| + \delta_{\mu,t})^p \int_0^{2\pi} \left|\left(|Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}}\right) + e^{i\beta}\left(|P'(e^{i\theta})| + \frac{mnt}{1+\delta_{\mu,t}}\right)\right|^p d\beta.$$

Again, integrating both sides with respect to θ from 0 to 2π, we obtain

$$\int_0^{2\pi} \left\{\left(|Q'(e^{i\theta})| - \frac{mnt}{1+\delta_{\mu,t}}\right) + |\alpha|\left(|P'(e^{i\theta})| + \frac{mnt}{1+\delta_{\mu,t}}\right)\right\}^p d\theta \int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}|^p d\beta \leq$$
$$\leq (|\alpha| + \delta_{\mu,t})^p \int_0^{2\pi}\int_0^{2\pi} \left|F(\theta) + e^{i\beta}G(\theta)\right|^p d\beta d\theta$$

where $F(\theta)$ and $G(\theta)$ are given by (16). Using this in inequality (18), we get

$$\int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}| d\beta \int_0^{2\pi} \left\{|D_\alpha[P](e^{i\theta})| + nmt\left(\frac{|\alpha| - \delta_{\mu,t}}{1+\delta_{\mu,t}}\right)\right\}^p d\theta \leq$$
$$\leq (|\alpha| + \delta_{\mu,t})^p \int_0^{2\pi}\int_0^{2\pi} |F(\theta) + e^{i\beta}G(\theta)|^p d\beta d\theta. \quad (19)$$

By using (17) in (19), we obtain for each $p > 0$ and $|\alpha| \geq 1$

$$\int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}| d\beta \int_0^{2\pi} \left\{|D_\alpha[P](e^{i\theta})| + nmt\left(\frac{|\alpha| - \delta_{\mu,t}}{1+\delta_{\mu,t}}\right)\right\}^p d\theta \leq$$
$$\leq (|\alpha| + \delta_{\mu,t})^p 2\pi n^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta.$$

Equivalently,

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \left\{|D_\alpha[P](e^{i\theta})| + nmt\left(\frac{|\alpha| - \delta_{\mu,t}}{1+\delta_{\mu,t}}\right)\right\}^p d\theta\right)^{1/p} \leq$$
$$\leq \frac{n(|\alpha| + \delta_{\mu,t})}{\left(\frac{1}{2\pi}\int_0^{2\pi} |\delta_{\mu,t} + e^{i\beta}| d\beta\right)^{1/p}} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right)^{1/p}$$

which immediately leads to (15) and this completes the proof of Theorem for $p > 0$. To obtain this result for $p = 0$, we simply make $p \to 0+$. □

References

1. Markov A. Sur une question posee par Mendeleieff. *Bull. Acad. Sci. de St. Petersburg* **62**, 1–24 (1889).

2. Milovanović G. V., Mitrinović D. S., Rassias Th. *Topics in polynomials: Extremal properties. Inequalituies, Zeros.* Singapore, World scientific (1994).

3. Schaeffer A. C. Inequalities of A. Markoff and S. N. Bernstein for polynomials and related functions. *Bull. Amer. Math. Soc.* **47**, 565–579 (1941).

4. Zygmund A. A remark on conjugate series. *Proc. London Math. Soc.* **34**, 392–400 (1932).

5. Arestov V. V. On integral inequalities for trigonometric polynomials and their derivatives. *Math. USSR-Izv.* **18**, 1–17 (1982). http://dx.doi.org/10.1070/IM1982v018n01ABEH001375 [Russ. ed.: Ob integral'nykh neravenstvakh dlia trigonometricheskikh mnogochlenov i ikh proizvodnykh. *Izv. Akad. Nauk SSSR Ser. Mat.* **45**, 3–22 (1981)].

6. De Bruijn N. G. Inequalities concerning polynomial in the complex domain. *Nederl. Akad. Wetensch. Proc.* **50**, 1265–1272 (1947).

7. Rahman Q. I., Schmeisser G. L_p inequalitites for polynomial. *J. Approx. Theory* **53**, 26–32 (1988). https://doi.org/10.1016/0021-9045(88)90073-1

8. Govil N. K., Rahman Q. I. Functions of exponential type not vanishing in a half-plane and related polynomials. *Trans. Amer. Math. Soc.* **137**, 501–517 (1969). https://doi.org/10.2307/1994818

9. Rather N. A. *Extremal properties and location of the zeros of polynomials.* PhD thesis. University of Kashmir (1998).

10. Marden M. *Geometry of polynomials.* In: Mathematical surveys, no. 3. Amer. Math. Soc. (1989).

11. Aziz A., Rather N. A. On an inequality concerning the polar derivative of a polynomial. *Proc. Indian Acad. Sci. (Math. Sci.)* **117**, 349–357 (2007). https://doi.org/10.1007/s12044-007-0030-0

12. Aziz A., Rather N. A., Aliya Q. L_q norm inequalities for the polar derivative of a polynomial. *Math. Ineq. and Appl.* **11**, 283–296 (2008).

13. Rather N. A. Some integral inequalities for the polar derivative of a polynomial. *Math. Balk.* **22**, 207–216 (2008).

14. Rather N. A. L_p inequalitites for the polar derivative of a polynomial. *J. Inequal. Pure and Appl. Math.* **9** (4), 103 (2008).

15. Rather N. A., Iqbal A., Hyun H. G. Integral inequalities for the polar derivative of a polynomial. *Non Linear Funct. Anal. Appl.* **23** (2), 381–393 (2018).

16. Gulzar S., Rather N. A. Inequalities concerning the polar derivative of a polynomial. *Bull. Malays. Math.Sci. Soc.* **40**, 1691–1700 (2017). https://doi.org/10.1007/s40840-015-0183-4

17. Aziz A., Rather N. A. L_p inequalities for polynomials. *Glas. Math.* **32**, 39–43 (1997).

18. Aziz A., Rather N. A. Some Zygmund type L_q inequalities for polynomials. *J. Math. Anal. Appl.* **289**, 14–29 (2004). https://doi.org/10.1016/S0022-247X(03)00530-4

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L_p-неравенства для полярной производной полинома лакунарного типа

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В настоящей работе мы распространяем неравенство относительно полярной производной многочлена в L_p-норме на класс лакунарных многочленов и тем самым получаем оценку, которая также зависит от некоторых коэффициентов многочлена.

Ключевые слова: L_p-неравенства, полярная производная, многочлены.

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